CORNER-COLLISION AND GRAZING-SLIDING

Practical examples of border-collision bifurcations

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Abstract: This chapter gives an overview of the main types of nonsmooth transitions which can be observed in piecewise smooth dynamical systems. Particular attention is given to those events involving interactions with the discontinuity boundary of fixed points of piecewise-smooth maps and limit cycles of piecewise-smooth flows. Strategies to classify these phenomena are discussed. It is shown that only few cases lead to maps which are locally piecewise linear to leading order. A nonlinear friction oscillator is used as a representative example to illustrate the main ideas introduced in the chapter.

Key words: Bifurcations, piecewise-smooth systems, friction oscillators

1. Introduction

Vibro-impacting systems can exhibit a multitude of different nonsmooth bifurcation phenomena [1]. Recently, for example, self-excited vibrating systems with dry-friction were studied by [2]. A route to chaos is reported where a period-doubling cascade is abruptly terminated by an outburst of chaotic behaviour due to the transition from slip to stick-slip motion, that cannot be explained by smooth bifurcation theory. Often it is conjectured that these and similar observations in the literature can be explained by the theory of so-called border collision bifurcation, which applies to discrete-time maps which are to lowest-order piecewise linear [3]. In a few examples, border collisions have indeed been shown to organise the dynamics, e.g. in DC/DC converters in Power Electronics [4], but in general it is hard to analytically derive the border-collision maps direct from the nonsmooth ordinary differential equations.

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In fact, analytically there is a fundamental problem recently outlined in [5], [6]. Here, the theory of discontinuity mappings shows that periodic orbits hitting tangentially (“grazing”) a discontinuity set in a nonsmooth continuous time system do not generically lead to maps which are locally piecewise linear. Instead the maps have either square-root or $O(3/2)$ singularities. It is now becoming clear that border-collision of piecewise linear maps therefore is not the whole story.

Nevertheless, in this chapter, we will present an overview of our recent work, showing two cases where border-collisions of piecewise linear maps can rigorously be derived and shown to organise the occurrence of chaotic dynamics. These are so-called the corner-collision bifurcation and the grazing-sliding bifurcation, the former of which occurs when a switching boundary is itself nonsmooth, and the latter is one of the ways in which pure slip motion can transform into stick-slip. As a representative example, we will discuss in detail the latter bifurcation occurring in the friction oscillator studied in [2].

2. Bifurcations of Nonsmooth Systems

Piecewise smooth (PWS) dynamical systems can exhibit most of the standard bifurcations found in smooth systems, for instance fold or period-doublings. In addition to these, there are also some novel transitions which are unique to PWS systems, which were given the name $C$-bifurcations in the Russian literature [7]. A $C$-bifurcation in this sense is any transition which can be explained in terms of interactions between invariant sets and switching surfaces in phase space. Note that according to this definition a $C$-bifurcation does not necessarily imply the onset of a topologically non-equivalent phase portrait at the bifurcation point.

We focus on two types of $C$-bifurcations: (i) Border-Collisions of fixed points in maps; (ii) Grazing Bifurcations of limit cycles in flows. Both cases are characterised by the same phase space topology close to the bifurcation point. Namely, the map or flow exhibiting the bifurcation is defined over a region $D \subset R^n$ of phase space which is chosen so that, by an appropriate choice of local coordinates, the map or flow under investigation can be described as:

$$\Phi[x] = \begin{cases} g_1(x, \mu) & \text{if } H(x) < 0 \\ g_2(x, \mu) & \text{if } H(x) > 0 \end{cases}$$

where $\Phi$ is a differential or finite difference operator, $H(x)=0$ defines a smooth boundary $\Sigma$ which separates $D$ into two regions denoted by $G_1$ and $G_2$, i.e. $\Sigma := \{x \in D \mid H(x) = 0\}, G_1 := \{x \in D \mid H(x) > 0\}, G_2 := \{x \in D \mid H(x) < 0\}$. We assume that $g_1(x, \mu) \in C^k$ if $x \in G_1$, $g_2(x, \mu) \in C^k$ if $x \in G_2$ and $g_1(x, \mu) = g_2(x, \mu)$ when $x \in \Sigma$, i.e. the map or flow is smooth to order $k$ in each of the subregion $G_1$ and $G_2$ while is continuous but