NEAR-RINGS, COHOMOLOGY AND EXTENSIONS

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Abstract  After historical considerations on the cohomology of groups and near-rings and extensions of near-rings, we analyze some near-rings playing a role in constructing the cohomology of groups. Then the notion of pseudo-modules does appear naturally and it is presented.

1. Some history and non only it

In 1960-1962, by a series of papers, A. Fröhlich (see references) introduced the derived functors and satellites in noncommutative homological algebra. These functors appeared in connection with the notion of group pairs, considered also by A. Fröhlich.

If N is a distributively generated (by S) left near-ring, then an \((N,S)\)-group \(A\) is a group \((A,+)\) together with a multiplication \(N \times A \to A\), such that: 
\[(x + y)a = xa + ya, \quad (xy)a = x(ya), \quad s(a + b) = sa + sb, \quad \text{for all} \quad x,y \in N, \quad a,b \in A \quad \text{and} \quad s \in S.\]

Let \(A\) be an \((N,S)\)-group and \(A'\) be a central \((N,S)\)-subgroup of it. Then \(A/A'\) is called a pair of \((N,S)\)-groups or a group pair.

A homomorphism of group pairs, \(f/f': A/A' \to B/B'\), is given by a commutative diagram

\[
\begin{array}{ccc}
A' & \xrightarrow{f'} & B' \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array}
\]

where the vertical arrows are the inclusions. Usually, \(f\) has also the property: \(f(xa) = xf(a)\), for all \(x\) in \(N\), \(a\) in \(A\). We note that \(f'\) is the restriction of \(f\) to \(A'\) and \(\text{Im} f' \subseteq B'\).

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One may consider $\text{MAP}(A/A', B/B')$, the set of all mappings $f: A \to B$ with the restriction to $A'$, $f': A' \to B'$, making commutative a diagram like (1.1). This is a group under pointwise addition, which is a subgroup in $\text{MAP}(A, B)$. The set of homomorphisms of pairs is not a subgroup of $\text{MAP}(A/A', B/B')$ but generates one, namely the subgroup $\text{HOM}_N(A/A', B/B')$. This is also a subgroup of $\text{HOM}_N(A, B)$, the subgroup generated by $\text{Hom}_N(A, B)$ in $\text{MAP}(A, B)$.

In $\text{MAP}(A, A)$, the set of all distributive elements, namely $\text{End}(A)$, generates a subnear-ring $E(A)$. The same is true for the group $B$.

Then, $E(A)$ acts on the right hand on $\text{HOM}(A, B)$ as well as $E_N(A)$ acts on the right hand on $\text{HOM}_N(A, B)$ or $\text{HOM}_N(A/A', B/B')$, while $E(B)$ acts on the left.

Indeed, if we consider $\alpha, \alpha' \in E(A)$, $\beta, \beta' \in \text{HOM}(A, B)$, then, by taking usual mapping composition $\beta \alpha : A \to B$, $\beta \alpha (a) := \beta (\alpha (a))$, for all $a \in A$, we have: $(\beta \alpha) \alpha' = \beta (\alpha \alpha')$ and $(\beta + \beta') \alpha = \beta \alpha + \beta' \alpha$.

Now, if $\gamma \in E(B)$, then $\gamma \beta \in \text{HOM}(A, B)$ and this is a right action of $E(B)$ on the group $\text{HOM}(A, B)$.

The same will happen for $N$-groups.

Then both $\text{HOM}_N(A, B)$ and $\text{HOM}_N(A/A', B/B')$ are right $E_N(A)$-groups and left $E_N(B)$-groups.

Fröhlich used these algebraic structures for constructing a theory of derived functors and satellites for $N$-group pairs.

In the introduction of the first paper [4], A. Fröhlich wrote:

"In spite of this similarity to the theory for modules, there are natural fundamental differences. In the first place, the sum of homomorphism of noncommutative groups is no longer a homomorphism; this is essentially the reason why the derived functors and satellites fail to be additive though they still retain the property of "preserving" null mappings". In fact, for compensating this, Fröhlich has introduced group pairs.

The second attempt to study cohomology of near-rings was made by Hans Lausch ([10] and [11]), who considered extensions of near-rings in connexion with the extension of groups (Baer, Schreier, MacLane) and rings (Everett). More precisely, the extension of a zero-ring by a distributively generated near-ring is built in his two papers.

In this respect, the concept of a trimodule over a d.g. near-ring $(R, S)$ is introduced. This is an abelian group with three external operations with "scalars" in $R$, satisfying relatively natural conditions, if we think of the constructions of such extensions for modules over rings, where the concept of bimodule has been used by Eilenberg and MacLane.