MOSAICS AND WATERSHEDS

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Abstract We investigate the effectiveness of the divide set produced by watershed algorithms. We introduce the mosaic to retrieve the altitude of points along the divide set. A desirable property is that, when two minima are separated by a crest in the original image, they are still separated by a crest of the same altitude in the mosaic. Our main result states that this is the case *if and only if* the mosaic is obtained through a topological thinning.

Keywords: segmentation, graph, mosaic, (topological) watershed, separation

Introduction

The watershed transform, introduced by S. Beucher and C. Lantuéjoul [4] for image segmentation, is now used as a fundamental step in many powerful segmentation procedures [8]. Watershed algorithms build a partition of the space by associating an influence zone to each minimum of the image, and by producing (in their "dividing" variant) a divide set which separates those influence zones; that is to say, they "extend" the minima.

In order to evaluate the effectiveness of the separation, we have to consider the altitude of points along the divide set. We call the greyscale image thus obtained a mosaic. The goal of this paper is to examine some properties of mosaics related to image segmentation. We say informally that a watershed algorithm produces a "separation" if the minima of the mosaic are of the same altitude as the ones of the original image and if, when two minima are separated by a crest in the original image, they are still separated by a crest of the same altitude in the mosaic. The formal definition relies on the altitude of the lowest pass which separates two minima, named pass value (see also [1, 9, 10]). Our main result states that a mosaic is a separation *if and only if* it is obtained through a topological thinning [5].

C. Ronse et al. (eds.), Mathematical Morphology: 40 Years On, 187–196.

1. Basic notions and notations

Many fundamental notions related to watersheds in discrete spaces can be expressed in the framework of graphs. Let E be a finite set of vertices (or points), and let $\mathcal{P}(E)$ denote the set of all subsets of E. Throughout this paper, Γ denotes a binary relation on E, which is reflexive ($(x,x) \in \Gamma$) and symmetric ($(x,y) \in \Gamma \Leftrightarrow (y,x) \in \Gamma$). We say that the pair (E,Γ) is a *graph*. We also denote by Γ the map from E to $\mathcal{P}(E)$ such that, for all $x \in E$, $\Gamma(x) = \{y \in E | (x,y) \in \Gamma\}$. For any point x, the set $\Gamma(x)$ is called the *neighborhood of* x. If $y \in \Gamma(x)$ then we say that x and y are *adjacent*.

Let $X \subseteq E$. We denote by $\overline{X}$ the complement of X in E. Let $x_0, x_n \in X$. A *path from* x_0 *to* x_n *in* X is a sequence $\pi = (x_0, x_1, \ldots, x_n)$ of points of X such that $x_{i+1} \in \Gamma(x_i)$, with $i = 0 \ldots n-1$. Let $x, y \in X$, we say that x and y are *linked for* X if there exists a path from x to y in X. We say that X is *connected* if any x and y in X are linked for X. We say that $Y \subseteq E$ is a *connected component of* X if $Y \subseteq X$, Y is connected, and Y is maximal for these two properties (*i.e.*, $Y = Z$ whenever $Y \subseteq Z \subseteq X$ and Z is connected). In the following, we assume that the graph (E,Γ) is connected, that is, E is made of exactly one connected component.

We denote by $\mathcal{F}(E)$ the set composed of all maps from E to $\mathbb{Z}$. A map $F \in \mathcal{F}(E)$ is also called an *image*, and if $x \in E$, $F(x)$ is called the *altitude of* x *(for* F*)*. Let $F \in \mathcal{F}(E)$. We write $F_k = \{x \in E | F(x) \geq k\}$ with $k \in \mathbb{Z}$, F_k is called an *upper section of* F, and $\overline{F_k}$ is called a *lower section of* F. A non-empty connected component of a lower section $\overline{F_k}$ is called a *(level* k*) lower-component of* F. A level k lower-component of F that does not contain a level $(k-1)$ lower-component of F is called a *(regional) minimum of* F. We denote by $\mathcal{M}(F)$ the set of minima of F.

A subset X of E is *flat for* F if any two points x, y of X are such that $F(x) = F(y)$. If X is flat for F, we denote by $F(X)$ the altitude of any point of X for F.

2. Minima extensions and mosaics

The result of most of the watershed algorithms is to associate an influence zone to each minimum of the image. We formalize this through the definition of a minima extension (see figure 1).

DEFINITION 1 *Let* X *be a subset of* E*, and let* $F \in \mathcal{F}(E)$*. We say that* X *is a* minima extension of F *if:*
- each connected component of X *contains one and only one minimum of* F*, and*
- each regional minimum of F *is included in a connected component of* X*.*
The complement of a minima extension of F *in* E *is called a* divide set of F.