REGULAR METRIC: DEFINITION AND CHARACTERIZATION IN THE DISCRETE PLANE

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Abstract We say that a metric space is regular if a straight-line (in the metric space sense) passing through the center of a sphere has at least two diametrically opposite points. The normed vector spaces have this property. Nevertheless, this property might not be satisfied in some metric spaces. In this work, we give a characterization of an integer-valued translation-invariant regular metric defined on the discrete plane, in terms of a symmetric subset B that induces through a recursive Minkowski sum, a chain of subsets that are morphologically closed with respect to B.

Keywords: Mathematical Morphology, symmetric subset, ball, lower regularity, upper regularity, regular metric space, integer-valued metric, translation-invariant metric, triangle inequality, recursive Minkowski sum, morphological closed subset, discrete plane, computational geometry, discrete geometry, digital geometry

Introduction

The continuous plane or, more precisely, the two-dimensional Euclidean vector space, has good geometrical properties. For example, in such space, a closed ball is included in another one only if the radius of the latter is greater than or equal to the sum of the distance between their centers and the radius of the former. Furthermore, in such space, two closed balls intersect each other if the sum of their radii is greater than or equal to the distance between their centers. Nevertheless, not all metric spaces have these properties.

In the first part of this work, we clarify the concept of regular metric space in which the above two geometrical properties are satisfied.

We say that a metric space is regular if its metric satisfies three regularity axioms or equivalently if a straight-line (in the metric space sense) passing through the center of a sphere has at least two diametrically opposite points. The Minkowski spaces (i.e., finite dimensional normed vector spaces) have this property.

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This regularity is generally lost when a metric on the continuous plane is restricted to the discrete plane, as it is the case of the Euclidean metric.

In the second part of this work, we study the characterization of the integer-valued translation-invariant regular metrics on the discrete plane in terms of some appropriate symmetric subsets.

We show that every such metric can be characterized in terms of a symmetric subset B that induces through a recursive Minkowski sum, a chain of subsets that are morphologically closed with respect to B.

Our characterization shows how to construct an integer-valued translation-invariant regular metric on the discrete plane.

This is an important issue in digital image analysis since the image domains are then discrete. In the sixties, Rosenfeld and Pfaltz [7] have already introduced a metric property and have used it to describe algorithms for computing some distance functions by performing repeated local operations. It appears that their property is precisely a necessary condition for a metric to be regular.

We came across the regularity property for a metric while we were trying to prove the one-pixel width of the skeleton of the "expanded" subsets proposed in [1]. Actually, what we needed at that time was just a "lower" regularity.

In one dimension, we observed that the (discrete) convexity is not a necessary condition to have the morphological closure property, so it was useless to solve our problem.

For the sake of simplicity of the presentation, in this work, we limit ourselves to the class of integer-valued metrics. More precisely, we consider the class of metrics that are mappings from the discrete plane *onto* the set of integers. This should not be a serious limitation because on the discrete plane the metrics assume only a countable number of values.

In Section 1, we give an axiomatic definition of regular metric spaces and we show that of the three axioms only two are sufficient to define the metric regularity. In order to get the regular metric characterization in the last section, we give in Section 2, independently of the metric definition, a definition of balls based on the notions of set translation and set transposition. In the same section, we recall the notions of recursive Minkowski sum, generated balls and border. In Section 3, we study the properties of the balls of a regular metric space. Conversely, in Section 4, we study the properties of the metric spaces constructed from the symmetric balls having a morphological closure property. Finally, in Section 5, we show the existence of a bijection between the set of integer-valued translation-invariant regular metrics defined on the discrete plane and the set of the symmetric balls satisfying the morphological closure property.