# GREY-WEIGHTED, ULTRAMETRIC AND LEXICOGRAPHIC DISTANCES

Fernand Meyer
*Centre de Morphologie Mathématique, École des Mines de Paris*
*35 rue Saint-Honoré, 77305 Fontainebleau, France*
fernand.meyer@ensmp.fr

**Abstract** Shortest distances, grey weighted distances and ultrametric distance are classically used in mathematical morphology. We introduce a lexicographic distance, for which any segmentation with markers becomes a Voronoï tessellation.

**Keywords:** Grey-weighted distances, ultrametric distances, lexicographic distances, marker segmentation, path algebra

## 1. Introduction

Mathematical morphology makes a great use of distances. The classical distance $d(x, y)$ between two pixels $x$ and $y$ is defined as the length of the shortest path linking these two pixels. If the family of admissible paths is constrained to remain within a set $Y$ or on the contrary to miss a set $Z$ we obtain geodesic distances. Grey weighted distances are obtained by assigning a weight to each edge of a graph. The length of a path being the sum of the weights of all its edges. Among them are the Chamfer distances [2], approximating the Euclidean distances on a grid and topographic distances for the construction of the watershed line [10, 7].

Ultrametric distances govern morphological segmentation. This is due to the fact that watershed segmentation is linked to the flooding of topographic surfaces: the minimal level of flooding for which two pixels belong to a same lake precisely is an ultrametric distance. The key contribution of this paper is the introduction of a lexicographic distance, for which a segmentation with markers becomes a Voronoï tessellation of SKIZ of the markers.

The paper is organized as follows. Restricting ourselves to distances defined on graphs, we present the various distances encountered in morphology. We then analyze the segmentation with markers and show that the flooding ultrametric distance is myopic: only a lexicographic distance has sufficient discriminant power for correctly describing the segmentation with markers as



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a SKIZ of the markers. In the last part we introduce the "path algebra", which unifies all shortest distance algorithms whatever their type.

# 2. Graphs and distances

## Graphs encountered in morphology

Graphs are the good framework for dealing with distances. A non oriented graph $G = [X, E]$ is a collection of a set $X$ whose elements are called vertices or nodes and of a set $E$ whose elements $e_{ij} = (i, j) \in E$ are pairs of vertices called edges. Two edges that share one or several nodes are said to be *adjacent*. *A path* of length $n$ is a sequence of $n$ edges $L = \{e_{12}, u_{23,} \ldots, e_{n-1n}\}$, such that successive edges are adjacent. Any partition $\mathcal{A}$ for which a dissimilarity between adjacent regions has been defined can be represented as a region adjacency graph $G = (X, E)$, where $X$ is the set of nodes and $E$ is the set of edges. The nodes represent regions of the partition. Adjacent regions $i$ and $j$ are linked by an edge $e_{ij} = (i, j)$ with a weight $s_{ij}$ expressing the dissimilarity between them. The adjacency matrix $A = (a_{ij})$ of the graph is defined by $a_{ij} = s_{ij}$ if $(i, j) \in E$ and $a_{ij} = \infty$ if not. By convention $a_{ii} = 0$ as $a_{ii}$ represents the dissimilarity between $i$ and $i$ itself.

In the simplest case, $A$ represents the pixels of the image, the edges linking neighboring pixels. In case of a topographic surface, the watershed graph is obtained by taking the catchment basins as nodes, the dissimilarity between two adjacent basins being the altitude of the pass separating them.

## Grey weighted distances

The "weighted length"of a path is defined as the sum of the weights of its edges. If the weight of an edge is equal to its length, we obtain the length of the path. The distance $d(x, y)$ between two nodes $x$ and $y$ is the minimal length of all paths between $x$ and $y$. If there is no path between them, the distance is equal to $\infty$. In fig.1A the shortest path between $x$ and $y$ is a bold line and has a length of 4.

This classical distance is well known and allows to define the distance between a pixel and a set, the Hausdorff distance between sets, distance functions, crest and saddle points, ultimate erosions, skeletons, Voronoï tesselations, extremities of particles etc. A recent review on the use of distance functions in mathematical morphology may be found in [11]. Grey weighted distances [12] have been used for finding the shortest paths in an image, the weight of the edges being an increasing function of the grey tones of their extremities. This allows doing virtual endoscopy [3] or detecting fibers on a noisy background [13]. The watershed line itself is the skeleton of influence of the regional min-