SEMIDISCRETE AND DISCRETE WELL-POSEDNESS OF SHOCK FILTERING

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Abstract While shock filters are popular morphological image enhancement methods, no well-posedness theory is available for their corresponding partial differential equations (PDEs). By analysing the dynamical system of ordinary differential equations that results from a space discretisation of a PDE for 1-D shock filtering, we derive an analytical solution and prove well-posedness. Finally we show that the results carry over to the fully discrete case when an explicit time discretisation is applied.

Keywords: Shock filters, analytical solution, well-posedness, dynamical systems.

1. Introduction

Shock filters are morphological image enhancement methods where dilation is performed around maxima and erosion around minima. Iterating this process leads to a segmentation with piecewise constant segments that are separated by discontinuities, so-called shocks. This makes shock filtering attractive for a number of applications where edge sharpening and a piecewise constant segmentation is desired.

In 1975 the first shock filters have been formulated by Kramer and Bruckner in a fully discrete manner [6], while first continuous formulations by means of partial differential equations (PDEs) have been developed in 1990 by Osher and Rudin [8]. The relation of these methods to the discrete Kramer–Bruckner filter became clear several years later [4, 12]. PDE-based shock filters have been investigated in a number of papers. Many of them proposed modifications with higher robustness under noise [1, 3, 5, 7, 12], but also coherence-enhancing shock filters [14] and numerical schemes have been studied [11].

Let us consider some continuous d-dimensional initial image $f : \mathbb{R}^d \to \mathbb{R}$. In the simplest case of a PDE-based shock filter [8], one obtains a filtered

C. Ronse et al. (eds.), Mathematical Morphology: 40 Years On, 311–320.

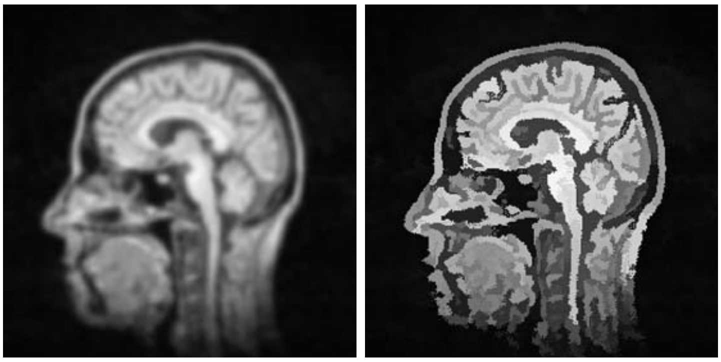

Figure 1. LEFT: Original image. RIGHT: After applying the Osher–Rudin shock filter.

version $u(x,t)$ of $f(x)$ by solving the evolution equation

$$\partial_t u = -\mathrm{sgn}(\Delta u)\,|\nabla u| \qquad (t \geq 0)$$

with f as initial condition, i. e. $u(0,x) = f(x)$. Experimentally one observes that within finite "evolution time" t, a piecewise constant, segmentation-like result is obtained (see Fig. 1).

Specialising to the one-dimensional case, we obtain

$$\partial_t u = -\mathrm{sgn}(\partial_{xx} u)\,|\partial_x u| = \begin{cases} |\partial_x u|\,, & \partial_{xx} u < 0, \\ -\,|\partial_x u|\,, & \partial_{xx} u > 0, \\ 0, & \partial_{xx} u = 0. \end{cases} \tag{1}$$

It is clearly visible that this filter performs dilation $\partial_t u = |\partial_x u|$ in concave segments of u, while in convex parts the erosion process $\partial_t u = -|\partial_x u|$ takes place. The time t specifies the radius of the interval (a 1-D disk) $[-t, t]$ as structuring element. For a derivation of these PDE formulations for classical morphological operations, see e.g. [2].

While there is clear experimental evidence that shock filtering is a useful operation, no analytical solutions and well-posedness results are available for PDE-based shock filters. In general this problem is considered to be too difficult, since shock filters have some connections to classical ill-posed problems such as backward diffusion [8, 7].

The goal of the present paper is to we show that it is possible to establish analytical solutions and well-posedness as soon as we study the *semidiscrete* case with a spatial discretisation and a continuous time parameter t. This case is of great practical relevance, since digital images already induce a natural space discretisation. For the sake of simplicity we restrict ourselves to the 1-D case. We also show that these results carry over to the fully discrete case with an explicit (Euler forward) time discretisation.