A VARIATIONAL FORMULATION OF PDE'S FOR DILATIONS AND LEVELINGS

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Abstract Partial differential equations (PDEs) have become very useful modeling and computational tools for many problems in image processing and computer vision related to multiscale analysis and optimization using variational calculus. In previous works, the basic continuous-scale morphological operators have been modeled by nonlinear geometric evolution PDEs. However, these lacked a variational interpretation. In this paper we contribute such a variational formulation and show that the PDEs generating multiscale dilations and erosions can be derived as gradient flows of variational problems with nonlinear constraints. We also extend the variational approach to more advanced object-oriented morphological filters by showing that levelings and the PDE that generates them result from minimizing a mean absolute error functional with local sup-inf constraints.

Keywords: scale-spaces, PDEs, variational methods, morphology.

1. Introduction

Partial differential equations have a become a powerful set of tools in image processing and computer vision for modeling numerous problems that are related to multiscale analysis. They need continuous mahematics such as differential geometry and variational calculus and can benefit from concepts inspired by mathematical physics. The most investigated partial differential equation (PDE) in imaging and vision is the linear isotropic heat diffusion PDE because it can model the Gaussian scale-space, i.e. its solution holds all multiscale linear convolutions of an initial image with Gaussians whose scale parameter is proportional to their variance. In addition, to its scale-space interpretation, the linear heat PDE can also be derived from a variational problem. Specifically, if we attempt to evolve an initial image into a smoother version by minimizing the L_2 norm of the gradient magnitude, then the PDE that results as the gradient descent flow to reach the minimizer is identical to the linear heat PDE. All

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the above ideas are well-known and can be found in numerous books dealing with classic aspects of PDEs and variational calculus both from the viewpoint of mathematical physics, e.g. [5], as well as from the viewpoint of image analysis, e.g. [12, 6, 14].

In the early 1990s, inspired by the modeling of the Gaussian scale-space via the linear heat diffusion PDE, three teams of researchers (Alvarez, Guichard, Lions & Morel [1], Brockett & Maragos [3, 4], and Boomgaard & Smeulders [19]) independently published nonlinear PDEs that model various morphological scale-spaces. Refinements of the above works for PDEs modeling multiscale morphology followed in [8, 7, 6]. However, in none of the previous works the PDEs modeling morphological scale-spaces were also given a direct variational interpretation. There have been only two indirect exceptions: i) Heijmans & Maragos [7] unified the morphological PDEs using Legendre-Fenchel 'slope' transforms, which are related to Hamilton-Jacobi theory and this in turn is related to variational calculus. ii) Inspired by the level sets methodology [13], it has been shown in [2, 15] that binary image dilations or erosions can be modeled as curve evolution with constant (± 1) normal speed. The PDE of this curve evolution results as the gradient flow for evolving the curve by maximizing or minimizing the rate of change of the enclosed area; e.g. see [17] where volumetric extensions of this idea are also derived. Our work herein is closer to [17].

In this paper we contribute a new formulation and interpretation of the PDEs modeling multiscale dilations and erosions by showing that they result as gradient flows of optimization problems where the volume under the graph of the image is maximized or minimized subject to some nonlinear constraints. Further, we extend this new variational interpretation to more complex morphological filters that are based on global constraints, such as the levelings [10, 11, 9].

2. Background

Variational Calculus and Scale-Spaces

A standard variational problem is to find a function $u = u(x, y)$ that minimizes the 'energy' functional

$$J[u] = \int\int F(x, y, u, u_x, u_y)dxdy \tag{1}$$

usually subject to natural boundary conditions, where F is a second-order continuously differentiable function. A necessary condition satisfied by an extremal function u is the Euler-Langange PDE $[F]_u = 0$, where $[F]_u$ is the Euler (variational) derivative of F w.r.t. u. In general, to reach the extremal function that minimizes J, we can set up a gradient steepest descent proce-