Chapter 5

The Algebra of Precisely Predictable Observables

5.0 Introduction

In this chapter we will start by discussing a precise theorem giving a necessary and sufficient condition for smoothness of the (inverse) Heisenberg transform, with some “framing conditions” added. Note, the symbol classes $\psi_{c_m}$ carry a “topology” (in fact, a Frechet topology), defined by the sup norms$^1$

$$\|a\|_{jl} = \sum_{|\theta|=j, |\iota|=l} \| \langle x \rangle^{-m_2+j} \langle \xi \rangle^{-m_1+l} a^{(t)}(x, \xi) \|_{L^\infty(\mathbb{R}^6)}, j, l = 0, 1, 2, \ldots,$$

where, as usual $\|b(x, \xi)\|_{\mathbb{R}^6} = \sup_{x, \xi \in \mathbb{R}^3} |b(x, \xi)|$. This allows a definition of differentiability of a symbol $a_t(x, \xi)$ for a parameter $t$: We shall say that

$$a_t = a_t(x, \xi) \text{ belongs to } C^\infty(\mathbb{R}, \psi_{c_m}) \text{ (or that the symbol } a_t \text{ depends smoothly on } t) \text{ if } a_t(x, \xi) \in C^\infty(\mathbb{R} \times \mathbb{R}^6) \text{ and if, in addition, the time-derivatives } \dot{a}_t, \ddot{a}_t, \ldots, \partial^j_t a_t, \ldots \text{ all exists in all of the above norms } (5.0.1).$$

Then also the $\psi$do $A_t = a_t(x, D)$ will be called a smooth functions of $t$ (within the space $Op\psi_{c_m}$).

We return to sec.4.2, and its assumptions there: time-dependent potentials $A_j, V$ satisfying cdn.$(X)$ with all their time-derivatives, for all $t$. We have the

$^1$In view of the results presented in ch.2 it is possible to carry over this Frechet topology to the operator class $Op\psi_{c_m}$ by using the (countably many) norms $\|\langle x \rangle^{m_2-|\iota|} ad_x^{\theta} ad_D^{\iota} A(D) \langle x \rangle^{-m_1-|\theta|} \|_H$. In view of (2.1.7) [which expresses the symbol as a trace of the product of a fixed trace class operator and an operator of the form (2.1.5)] we then get an equivalent topology on $Op\psi_{c_m}$. 

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evolution operator $U(\tau, t)$ of the Dirac equation, an operator of order 0, - for
time-independent $A_j, V$ coinciding with $e^{-i(t-\tau)H}$ - and consider the (inverse)
Heisenberg representation $A \to A_t = U(0, t)AU(t, 0)$, for an unbounded operator
$A$ acting on a dense subdomain of $\mathcal{H}$. We still assume $A \in Op\psi_c$ to be a strictly
classical $\psi$do, and set $\tau = 0$.

In ch.4 - centering around relation (4.2.7) - we were developing a procedure
to determine $A_t$ for a given $A$ - assuming that (i) $A_t = a_t(x, D)$ still belongs
to $Op\psi_c$, while (ii) even $a_t \in \psi_{c_{m-e}}$. Actually, it was seen that these two
conditions alone imply that the symbol $a(x, \xi)$ of such an operator must “nearly”
commute with the symbol $h(0; x, \xi)$ of the Hamiltonian $H(t)$ at $t = 0$. More
precisely, there must be a decomposition $a = q + z$ - and, generally, $a_t = q_t + z_t$,
where $q_t(x, \xi)$ commutes with $h(t; x, \xi)$ [so, $q(x, \xi) = q(0, x, \xi)$ commutes with
$h(0; x, \xi)$], for all $x, \xi$ while $z = z_0$, $z_t \in \psi_{c_{m-e}}$.

Also, starting with an arbitrary given symbol $q \in \psi_{c_m}$ with $[h(0; x, \xi), q(x, \xi)] = 0$
for all $x, \xi$, we were attempting to construct a “correction symbol” $z(x, \xi) \in
\psi_{c_{m-e}}$ - and, more generally, continuations $q_t$ with $[q_t, h(t)] = 0$ and $z_t(x, \xi) \in
\psi_{c_{m-e}}$ with $z_0 = z$ such that $A = a(x, D)$ with $a = q + z$ has a smooth (inverse)
Heisenberg representation, given by $a_t(x, D)$ with $a_t = q_t + z_t$.

Our construction - so far - was not carried out completely, insofar as only $q_t$
and an approximative $z$ and $z_t$ were obtained. This approximation was seen to be
useful, however, insofar as it was correct “modulo lower order” - that is, its error
tends to get negligibly small as $|x| + |\xi| \to \infty$. And the usefulness of this was
confirmed, perhaps, since, among other facts, we were able to derive the classical
equations of motion from it - including motion of the spin as a classical magnetic
moment vector.

In the present chapter we will offer a mathematically complete theory, showing
that an iteration of our procedure can be designed which indeed will supply a
precise correction symbol $z \in \psi_{c_{m-e}}$ for every symbol $q \in \psi_{c_m}$ commuting with
the Hamiltonian symbol $h(t; x, \xi)$ at $t = 0$ such that indeed (1) $q$ and $z$ both will
have “extensions” $q_t \in \psi_{c_m}$ and $z_t \in \psi_{c_{m-e}}$, for all $t \in \mathbb{R}$, where $q_0 = q$, $z_0 = z$,
while $[h(t; x, \xi), q_t(x, \xi)] = 0 \forall x, \xi \in \mathbb{R}^3, t \in \mathbb{R}$; (2) $q_t \in C^\infty(\mathbb{R}, \psi_{c_m})$ and $z_t \in
C^\infty(\mathbb{R}, \psi_{c_{m-e}})$; (3) $A = q(x, D) + z(x, D)$ and $A_t = q_t(x, D) + z_t(x, D)$ satisfy
$A_t = U(0, t)AU(t, 0)$ - that is, $A_t$ is the (inverse) Heisenberg representation of $A$.

In sec.5.1, below, we will set up the precise class of operators for the above,
and then will state and prove the corresponding theorem. The main ingredient of
the proof will just be an iteration of the procedure in sec’s 4.2 and 4.4. But we
again will need detailed facts about symbol propagation. These will be discussed