Factorization and classical Darboux transformations

In this chapter we describe the algebraical factorization-based method to dress solutions of (1+1)-dimensional equations. We also show how the Darboux transformation (DT) theory appears in this framework.

First, in Sect. 2.1, we introduce the non-Abelian Bell polynomials and then generalize them in Sect. 2.2 to formulate in Sect. 2.3 a problem of factorization of a polynomial differential operator in the form of division by a monomial from the right and from the left. The relation between the factorization rules and the classical Darboux theorem [102] generalized in [314] is described in Sect. 2.4: the formalism produces a compact form of the DT for non-Abelian coefficients of linear operators, polynomial in a differentiation on a ring. Section 2.5 is devoted to a representation of the iterated DTs in terms of quasideterminants. As a highly nontrivial example of the iterated DT formalism, we describe positon solutions of the Korteweg–de Vries (KdV) equation discovered by Matveev [318, 319].

The growing interest in discrete models appeals to wider classes of symmetry structures of the corresponding nonlinear problems [149, 196, 255, 256, 339]. Very recently a suitable basis for new searches in the field of differential-difference and difference-difference equations was discovered [321] in the framework of the classical DT theory such that the difference operator is replaced by an arbitrary automorphism transformation. In Sect. 2.6 we present the dressing method via factorization for such a kind of generalizations. Like in the case of differential operators, this approach demonstrates links with the Hirota bilinearization method [260] and the factorization theory [271], with similar applications. We reformulate the Darboux covariance theorem from the paper of Matveev [321] and introduce a kind of difference Bell polynomials. These polynomials correspond naturally to the differential (generalized) Bell polynomials in their non-Abelian version of Sect. 2.2.

The joint covariance principle is formulated in Sect. 2.7 for Abelian and in Sect. 2.8 for non-Abelian differential rings. The same construction for a pair of difference equations is elaborated in Sect. 2.9. The form of the DT presented here allows us to develop a classification scheme with respect to the DTs in
connection with the generalized Bell polynomials [187, 260, 467]. If a pair of such operators determines the Lax equations, the joint covariance with respect to the DTs produces a symmetry for the compatibility condition [314, 324]. In Sects. 2.10 and 2.11 we illustrate the possibilities of the method by examples of specific nonlinear equations: the non-Abelian Hirota system [210] having promising applications [149], and the Nahm equations [344]. We introduce a lattice Lax pair for the Nahm equations which is covariant with respect to combined Darboux-gauge transformations that generate the dressing structure for the equations. Finally, in Sect. 2.12 we illustrate the formalism developed, solving a particular case of the Nahm equations.

2.1 Basic notations and auxiliary results.
Bell polynomials

Let $K$ be a differential ring of the zero characteristics with unit $e$ (i.e., unitary ring) and with an involution denoted by a superscript asterisk. The differentiation is denoted as $D$. The differentiation and the involution are agreed with operations in $K$:

1. $(a^*)^* = a, (a + b)^* = a^* + b^*, (ab)^* = b^*a^*$, $a, b \in K$.
2. $D(a + b) = Da + Db$, $D(ab) = (Da)b + aDb$.
3. $(Da)^* = -Da^*$.
4. Operators $D^n$ with different $n$ form a basis in a $K$-module $\text{Diff}(K)$ of differential operators. The subring of constants is $K_0$ and a multiplicative group of elements of $K$ is $\mathcal{G}$.
5. For any $s \in K$ there exists an element $\varphi \in K$ such that $D\varphi = s\varphi$; this also means the existence of a solution of the equation

$$D\phi = -\phi s,$$ (2.1)

owing to the involution properties.

There are lots of applications of the rings of square matrices in the theory of integrable nonlinear equations, as well as in classical and quantum linear problems. In this case matrices are parameterized by a variable $x$ and $D$ can be a derivative with respect to this variable or a combination of partial derivatives that satisfies conditions 1 and 2. If $D$ is the standard differentiation, then the involution (asterisk) may be the Hermitian conjugation. In the case of a commutator, the operator $D$ acts as $Da = [d, a]$ and $(Da)^* = -[d^*, a]$. Having in mind this or similar applications, we shall refer to the involution as conjugation. We do not restrict ourselves to the matrix-valued case; an appropriate operator ring is also suitable for our theory.

Below we introduce left and right non-Abelian Bell polynomials (see also [388]) and formulate the statements for them. The differential Bell polynomials are defined in Definition 2.1: