# 5

# Description of Probabilistic Models

Two basic approaches to description of probabilistic CI structures are dealt with in this chapter. The first one, which uses structural imsets, was already mentioned in Section 4.2. The second one, which uses supermodular functions, is closely related to the first one. It can also use imsets over $N$ to describe CI models over $N$ but the respective class of imsets and their interpretation are completely different. However, despite the formal difference, the approaches are equivalent. In fact, there exists a certain duality relation between these two methods: one approach is complementary to the other (see Section 5.4). The main result of the chapter says that every CI model induced by a probability measure with finite multiinformation can be described both by a structural imset and by a supermodular function.

## 5.1 Supermodular set functions

A real set function $m : \mathcal{P}(N) \to \mathbb{R}$ is called a *supermodular function over* $N$ if

$$m(U \cup V) + m(U \cap V) \geq m(U) + m(V) \qquad \text{for every} \;\; U, V \subseteq N. \tag{5.1}$$

The class of all supermodular functions on $\mathcal{P}(N)$ will be denoted by $\mathcal{K}(N)$. The definition can be formulated in several equivalent ways.

**Proposition 5.1.** A set function $m : \mathcal{P}(N) \to \mathbb{R}$ is supermodular iff any of the following three conditions holds:

(i) $\langle m, u \rangle \geq 0$ for every structural imset $u$ over $N$,
(ii) $\langle m, u \rangle \geq 0$ for every semi-elementary imset $u$ over $N$,
(iii) $\langle m, u \rangle \geq 0$ for every elementary imset $u \in \mathcal{E}(N)$.

*Proof.* Evidently (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (i) follows from the definition of a structural imset (see Section 4.2.3, p. 73) and the linearity of the scalar product. The condition (5.1) is equivalent to the requirement $\langle m, u_{\langle A,B|C \rangle} \rangle \geq 0$ for every $\langle A, B|C \rangle \in \mathcal{T}(N)$ which is nothing but (ii). $\square$

Further evident observation is as follows.

**Proposition 5.2.** The class of supermodular functions $\mathcal{K}(N)$ is a cone:

$$\forall\, m_1, m_2 \in \mathcal{K}(N) \quad \forall\, \alpha, \beta \geq 0 \quad \alpha \cdot m_1 + \beta \cdot m_2 \in \mathcal{K}(N)\,. \tag{5.2}$$

*Remark 5.1.* This is to warn the reader that a different terminology is used in game theory, where supermodular set functions are named either "convex set functions" [109] or even "convex games" [116]. I followed the terminology from game theory in some of my former publications [131, 137]. However, in order to avoid confusion with the usual meaning of the adjective "convex" in mathematics another common term "supermodular" is used in this book. As mentioned in [17] supermodular functions are also named "2-monotone Choquet capacities". △

Two kinds of "scalar product" equivalence for supermodular functions are introduced and distinguished below. The weaker one will be called qualitative and the stronger one quantitative.

### 5.1.1 Semi-graphoid produced by a supermodular function

One says that a disjoint triplet $\langle A, B|C\rangle \in \mathcal{T}(N)$ is *represented in a supermodular function* $m$ over $N$ and writes $A \perp\!\!\!\perp B\,|\,C\ [m]$ if $\langle m, u_{\langle A,B|C\rangle}\rangle = 0$. The class of represented triplets then defines the *model produced by* $m$

$$\mathcal{M}^m = \{\, \langle A, B|C\rangle \in \mathcal{T}(N)\,; \quad A \perp\!\!\!\perp B\,|\,C\ [m]\,\}\,.$$

Two supermodular functions over $N$ are *qualitatively equivalent* if they represent the same class of disjoint triplets over $N$.

*Remark 5.2.* This is to explain terminology. I usually say that a model is induced by a mathematical object over $N$ (see Section 2.2.1); for example, by a probability measure over $N$ or by a graph over $N$ (see Chapter 3). However, in this and subsequent chapters I need to distinguish between two different ways of ascribing formal independence models to imsets. Both ways appear to be equivalent as concerns the overall class of ascribed models (see Corollary 5.3). The problem is that some imsets (e.g. the zero imset or $u_{\langle a,b|\emptyset\rangle}$ if $N = \{a, b\}$) may be ascribed different models depending on the way of ascribing. To prevent misunderstanding I decided to emphasize the difference both in terminology (*induced* model versus *produced* model) and in notation ($\mathcal{M}_u$ versus $\mathcal{M}^m$). I regret to admit that the adjective "induced" was also used in a former research report [145] in connection with supermodular functions.

Note that the difference between both ways of ascribing formal independence models is also reflected in names of respective equivalences of mathematical objects. The corresponding equivalence of supermodular functions