6

Equivalence and Implication

This chapter deals with the equivalence and implication problems for structural imsets. First, the question of how to understand the concept of equivalence (and implication) is discussed and two basic types of equivalence are compared. The rest of the chapter is devoted to the stronger type of equivalence, called the *independence equivalence*, and to the respective implication between structural imsets. Two characterizations of independence implication, which are analogous to graphical characterizations of independence equivalence of graphs mentioned in Chapter 3, are given, and related implementation tasks are discussed.

6.1 Two concepts of equivalence

Basically, there are two different ways of defining the concept of equivalence for graphs in connection with classic graphical models; these ways appear to be equivalent in standard situations. The first option is *independence equivalence* which is the requirement that the induced formal independence models coincide. This type of equivalence is not related to a distribution framework.

The second option is *distribution equivalence*. Actually, one can introduce various kinds of distribution equivalence, that is, the conditions that the respective statistical models coincide. This type of equivalence of graphs is always understood relative to a distribution framework, that is, relative to a fixed comprehensive class Ψ of probability measures over N such that the considered statistical models are introduced as certain subsets of Ψ – see Section A.9.5 for more details. A basic form of a distribution equivalence is *Markov equivalence* which is the requirement that the classes of Markovian measures within Ψ coincide. However, one can take into consideration other forms of distribution equivalence. One of them could be *factorization equivalence*, that is, the condition that classes of factorizable measures with respect to considered graphs coincide - see Remarks 3.3 and 3.7. Another form of distribution equivalence is *parameterization equivalence* mentioned in Remark 6.1 below.

Clearly, because of the definition of a Markovian measure, independence equivalence implies Markov equivalence regardless of what is the considered distribution framework. The converse is true in the case of faithfulness (see Section 1.1, p. 3). It is easy to see that if a perfectly Markovian measure within a distribution framework Ψ exists for every graph (from the respective universum of graphs) then Markov equivalence relative to Ψ implies independence equivalence. This is the case of classic chain graphs relative to the class of discrete measures (see Section 3.3) and the case of alternative chain graphs relative to the class of regular Gaussian measures (see Section 3.5.5).

Nevertheless, Markov and independence equivalence coincide even under a weaker assumption that the considered class of measures is perfect for every graph (see Remark 3.2 on p. 45 for this concept). On the other hand, if the considered distribution framework is somehow limited then it may happen that independence and Markov equivalence differ as the next example shows.

Example 6.1. One can design a special distribution framework Ψ in such a way that Markov equivalence of undirected graphs relative to Ψ does not imply their independence equivalence. Consider a discrete distribution framework with prescribed one-dimensional marginals P_i on fixed measurable spaces $(\mathsf{X}_i, \mathcal{X}_i)$, $i \in N$ (see Section A.9.5). Assume that at least two prescribed marginals P_k are *collapsed measures*, by which is meant that $P_k(\mathsf{A}) \in \{0,1\}$ for every $\mathsf{A} \in \mathcal{X}_k$. Supposing P_k is collapsed for every $k \in M$, $M \subseteq N$ one can verify using Lemma A.6 that $a \perp\!\!\!\perp b \,|\, K \; [P]$ for every $P \in \Psi$ and $\langle a, b|K\rangle \in \mathcal{E}(N)$ such that $a \in M$. Consequently, every two undirected graphs over N which have the same induced subgraph for $N \setminus M$ can be shown to be Markov equivalent relative to Ψ. However, if the graphs differ they are not independence equivalent (see p. 46). $\diamondsuit$

Remark 6.1. The third type of distribution equivalence is *parameterization equivalence*. This approach is based on the following interpretation of some types of graphs, e.g., ancestral graphs [107] and joint-response chain graphs [28]. A specific distribution framework Ψ, usually the class of regular Gaussian measures over N, is considered. Every edge in a graph of the above-mentioned type represents a real parameter and every collection (= a vector) of edge-parameters determines a unique probability measure from Ψ factorized in a particular way. Every graph of considered type is then identified with the class of *parameterized measures* which often coincides with the class of Markovian measures within Ψ (e.g., in the case of maximal ancestral graphs [107] mentioned in Section 3.5.8). Two graphs can be called *parameterization equivalent* if their classes of parameterized measures coincide. Of course, parameterization equivalence substantially depends on the considered distribution framework and may not coincide with Markov equivalence; for example, in the case of general ancestral graphs [107]. This particular point of view motivates a general question of whether some structural imsets may lead to a specific way of parameterizing the corresponding class of Markovian distribution (see Direction 6 in Chapter 9). $\triangle$