7

The Problem of Representative Choice

This chapter deals with the problem of choice of a suitable representative within a class of independence equivalent structural imsets. It is an advanced subquestion of the general equivalence question mentioned in Section 1.1 studied in the universum of structural imsets; an analogous question has already been treated in graphical universa – see Chapter 3 (the concept of an essential graph and the concept of the largest chain graph). A few principles of representative choice are introduced and discussed in this chapter. Special attention is devoted to the representation of graphical models by structural imsets. Two auxiliary criteria for the choice of a representative are introduced and discussed. A dual way of describing structural independence models is mentioned in the last section of the chapter.

7.1 Baricentral imsets

An imset u over N is called *baricentral* if it has the form

$$u = \sum_{w\in\mathcal{E}(N),\, u\rightharpoonup w} w \quad \text{or equivalently,} \quad u = \frac{1}{2}\cdot \sum_{\langle a,b|K\rangle\in\mathcal{M}_u\cap\mathcal{T}_\epsilon(N)} u_{\langle a,b|K\rangle}\,. \quad (7.1)$$

Evidently, every elementary imset is baricentral and every baricentral imset u is a combinatorial imset with the degree $|\{w\in\mathcal{E}(N); u\rightharpoonup w\}|$. Moreover, the definition implies that every independence equivalence class of structural imsets contains exactly one baricentral imset. Nevertheless, a semi-elementary imset need not be baricentral. Given a semi-elementary imset $u_{\langle A,B|C\rangle}$ for $\langle A,B|C\rangle\in\mathcal{T}(N)$, the respective independence equivalent baricentral imset need not even be its multiple, despite the fact that the formulas

$$deg(u_{\langle A,B|C\rangle}) = |A|\cdot|B|$$
$$|\{w\in\mathcal{E}(N)\,;\, u_{\langle A,B|C\rangle}\rightharpoonup w\}| = |A|\cdot|B|\cdot 2^{|A|-1}\cdot 2^{|B|-1}\,,$$

suggest that it might be the case.

Example 7.1. There exists a semi-elementary imset v over $N = \{a, b, c, d\}$ such that no multiple $k \cdot v$, $k \in \mathbb{N}$ is a baricentral imset. Put $v = u_{\langle a,bcd|\emptyset\rangle}$ – see the left-hand picture of Figure 7.1. Then $u \rightharpoonup w \in \mathcal{E}(N)$ iff $w = u_{\langle a,e|K\rangle}$ where $e \in \{b, c, d\}$, $K \subseteq \{b, c, d\} \setminus \{e\}$ (cf. Lemma 2.2). The respective baricentral imset u is shown in the right-hand picture of Figure 7.1. Observe that $12 = deg(u) = 4 \cdot deg(v)$ but $u \neq 4 \cdot v$ since the level-degrees of u and v are not proportional: $deg(v, l) = 1$ for $l = 0, 1, 2$ while $deg(u, 0) = deg(u, 2) = 3$ and $deg(u, 1) = 6$. On the other hand, $u = 3 \cdot v + u_{\langle a,b|c\rangle} + u_{\langle a,c|d\rangle} + u_{\langle a,d|b\rangle}$. ♢

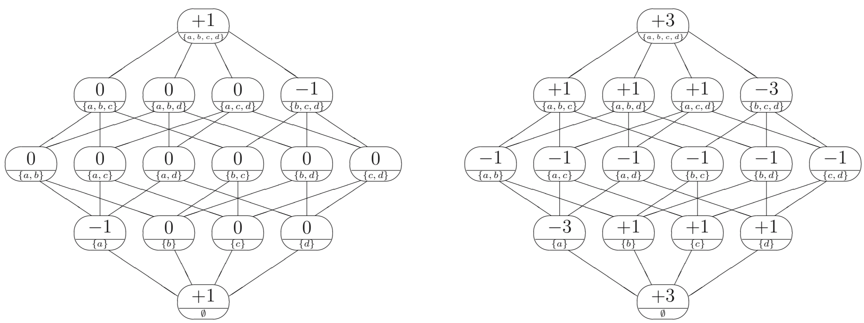

Fig. 7.1. Non-proportional respective semi-elementary and baricentral imsets.

The significance of baricentral imsets consists in the fact that testing independence implications between them is very simple.

Proposition 7.1. Let u, v be baricentral imsets over N. Then $u \rightharpoonup v$ iff $u - v \in \mathcal{C}(N)$.

Proof. If $u \rightharpoonup v$ then, for every $w \in \mathcal{E}(N)$, $v \rightharpoonup w$ implies $u \rightharpoonup w$ and (7.1) gives $u - v \in \mathcal{C}(N)$. The converse follows from Lemma 6.1. □

Note that testing combinatorial imsets is straightforward (see Remark 6.3). An analogous result holds if v is replaced by a semi-elementary imset (see Corollary 7.4 below). In particular, the induced model $\mathcal{M}_u$ can easily be identified on the basis of a baricentral imset u; that means no multiplication of u is needed as in the case of a general structural imset (see Section 4.4.1).

Remark 7.1. The terminology "baricentral imset" was inspired by a geometric idea that the class of those structural imsets that are i-implied by $v \in \mathcal{S}(N)$ is just the class of imsets belonging to the cone $con(\{w \in \mathcal{E}(N); v \rightharpoonup w\})$ (cf. Remark 6.2 on p. 114). Thus, a (minimal) balanced combination of all extreme imsets of this cone forms its "baricenter".