9

Open Problems

The goal of this chapter is to gather open problems and present a few topics omitted in the previous chapters. Open problems are classified according to their degrees of specificity in three categories. *Questions* are clear inquiries formulated as mathematical problems. Formal definitions of related concepts are given and the expected answer is yes or no. *Themes* (of research) are wider areas of mutually related problems. Their formulation is slightly less specific (but still in mathematical terms) and they may deserve some clarification of the involved concepts. *Directions* (of research) are wide groups of problems with a recognized common motivation source. They are formulated quite vaguely and may become a topic of research in forthcoming years. The secondary criterion of classification of open problems is their topic: the division of this chapter into sections was inspired by the motivational thoughts presented in Section 1.1.

9.1 Theoretical problems

In this section open problems concerning theoretical foundations are gathered. Some of them were already mentioned earlier. They are classified by their topics.

9.1.1 Miscellaneous topics

Multiinformation

There are some open problems related to the concept of multiinformation.

Question 1. Let P and Q be probability measures over N defined on the product of measurable spaces $(\mathsf{X}_N, \mathcal{X}_N) = \prod_{i \in N} (\mathsf{X}_i, \mathcal{X}_i)$ that have finite multiinformation (p. 24). Has their convex combination $\alpha \cdot P + (1-\alpha) \cdot Q$, $\alpha \in [0,1]$ finite multiinformation as well?

Question 2. Let $\mathcal{K}'(N)$ denote the conical closure of the set of multiinformation functions induced by discrete probability measures over N (see p. 11). Is $\mathcal{K}'(N)$ a rational polyhedral cone?

The answer to Question 2 is positive in the case $|N| \leq 3$; but I do not know the answer if $|N| = 4$. The significance of this question consists in the fact that discrete CI models can be characterized properly if the answer is positive.

Proposition 9.1. If the answer to Question 2 is positive then there exists a non-empty finite set $\mathbb{S} \subseteq \mathbb{Z}^{\mathcal{P}(N)} \setminus \{0\}$ such that every $s \in \mathbb{S}$ generates an extreme ray of $\mathcal{K}'(N)$ and $\mathcal{K}'(N) = con(\mathbb{S})$. Then the following conditions are equivalent for $\mathcal{M} \subseteq \mathcal{T}(N)$:

(i) $\mathcal{M}$ is a CI model induced by a discrete probability measure over N,
(ii) $\mathcal{M}$ is produced by an element of $\mathcal{K}'(N)$,
(iii) $\mathcal{M}$ has the form $\mathcal{M} = \bigcap_{t\in\mathbb{T}} \mathcal{M}^t$ where $\mathbb{T} \subseteq \mathbb{S}$.

Proof. Because $\mathcal{K}'(N) \subseteq \mathcal{K}_\ell(N)$ and $\mathcal{K}_\ell(N)$ is a pointed cone (Lemma 5.3), $\mathcal{K}'(N)$ is a pointed rational polyhedral cone. As mentioned in Section A.5.2, this implies that $\mathcal{K}'(N)$ has finitely many extreme rays and every extreme ray is generated by a non-zero integral vector. Moreover, $\mathcal{K}'(N)$ is their conical closure.

The implication (i) $\Rightarrow$ (ii) follows directly from Corollary 2.2. To prove (ii) $\Rightarrow$ (iii), suppose $\mathcal{M} = \mathcal{M}^m$ where $m = \sum_{s\in\mathbb{S}} \alpha_s \cdot s$ with $\alpha_s \geq 0$ and put $\mathbb{T} = \{t \in \mathbb{S}; \alpha_t > 0\}$. Using the fact $\mathcal{K}'(N) \subseteq \mathcal{K}_\ell(N)$ and Proposition 5.1(ii), we can derive $\mathcal{M} = \bigcap_{t\in\mathbb{T}} \mathcal{M}^t$. To prove (iii) $\Rightarrow$ (i), observe that every $s \in \mathbb{S}$ has the form $\alpha \cdot m_P$ for a discrete probability measure over N and $\alpha > 0$ (since s generates an extreme ray of $\mathcal{K}'(N)$). Thus, for every $s \in \mathbb{S} \cup \{0\}$, $\mathcal{M}^s$ is a discrete CI model and we can use Lemma 2.9 to derive (i). □

Moreover, it seems that if Question 2 has a positive answer then discrete CI models and inclusions between them can be characterized in terms of an arithmetic relationship between certain special imsets over N. What follows is more likely an intuitive plan than a list of verified claims. Roughly speaking, the plan is to repeat with the cone $\mathcal{K}'(N)$ something analogous to what was done with the cone $\mathcal{K}_\ell(N)$ in Chapters 5 and 6. However, it is quite possible that some of the steps indicated below cannot be made.

The first observation should be that the cone $\mathcal{K}'(N)$ has finitely many faces and each of them is generated by a finite subset $\mathbb{T} \subseteq \mathbb{S}$. The second step should be to establish a one-to-one correspondence between discrete CI models and faces of $\mathcal{K}'(N)$: every $\mathcal{M} \subseteq \mathcal{T}(N)$ is assigned the face $\{t \in \mathcal{K}'(N)\,;\, \mathcal{M} \subseteq \mathcal{M}^t\}$ and every $\mathbb{F} \subseteq \mathcal{K}'(N)$ is assigned the model $\bigcap_{t\in\mathbb{T}} \mathcal{M}^t$. The conjecture is that it should define the Galois connection in the sense of Section 5.4. The third possible step is to introduce a suitable pointed rational polyhedral cone $\mathcal{K}^*(N)$, which should correspond to the dual cone of $\mathcal{K}'(N)$. The cone $\mathcal{K}^*(N)$