5

Variance Error, Reproducing Kernels, and Orthonormal Bases

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5.1 Introduction

The preceding chapters have developed a general theory pertaining to rational orthonormal bases. They have also presented a general framework for addressing system identification problems and have then provided a link between the two by illustrating the advantages of parameterizing models for system identification purposes using rational orthonormal bases.

This chapter continues on these themes of exposing the links between rational orthonormal bases and the system identification problem. The main feature of this chapter is that it will progress to consider more general model structures, especially those not specifically formulated with respect to an orthonormal basis. At the same time that we generalize this aspect, we will focus another by considering only the noise-induced error. We will not discuss bias error, as it will be analysed in detail in the following chapters, and was also addressed in the previous one.

Many of the underlying ideas necessary for the developments here have already been introduced, but in the interests of the reader who is seeking to ‘dip into’ chapters individually, a brief introduction to essential background ideas and notation will be made at the expense of some slight repetition.

In précis then, the sole focus of this chapter is to examine and quantify noise-induced estimation errors in the frequency domain, and there are four essential points to be made here in relation to this.

1. The quantification of noise-induced error (variance error) is equivalent to the quantification of the reproducing kernel for a particular function space $X_n$;

2. This function space $X_n$ depends on the model structure, as well as on input and noise spectral densities. Hence, it is \textit{not} independent of model structure;
3. The quantification of the reproducing kernel, and hence the quantification of variance error, depends crucially on the construction of a rational orthonormal basis for the afore-mentioned function space $X_n$;

4. The variance error quantifications that result are quite different to certain pre-existing ones, while at the same time are often much more accurate. This latter point will be established here empirically by simulation example.

5.2 Motivation

To make the context and purpose of this chapter more concrete, and to also motivate the presented material, we begin with a simulation example to illustrate the practical impact of what will follow.

Consider a linear and time invariant system that in continuous time has transfer function

$$G(s) = \frac{0.0012(1 - 3.33s)^3}{(s + 0.9163)^2(s + 0.3567)^2(s + 0.2231)^3}.$$ 

Suppose that the response of this system is measured via the collection of input-output data sampled at 1 second intervals with zero-order-held inputs. This implies a discrete time representation

$$G(q) = \frac{-0.0177(q^2 - 2.7192q + 1.8489)(q - 4.1377) \times}{(q - 1.3298)(q + 0.4466)(q + 0.0463)} \frac{(q - 0.8)^3(q - 0.7)^2(q - 0.4)^2}{(q - 0.8)^3(q - 0.7)^2(q - 0.4)^2}.$$ 

Suppose further that we seek to estimate this transfer function on the basis of the observed input-output data via the use of the prediction error estimation methods described in the previous chapter. For this purpose, consider the case of a 7th order output-error model structure where we observe a length $N = 10,000$ sample record for which the output is corrupted by white Gaussian noise of variance $\sigma^2 = 0.01$, and with input that is a realization of a stationary Gaussian process with spectral density

$$\Phi_u(\omega) = \frac{1}{1.25 - \cos \omega}.$$ 

For this scenario, the sample mean square error in the estimation of the frequency response $G(e^{i\omega})$ and more then 10,000 estimation experiments with different input and noise realizations is used as an estimate of the variability $\text{Var}\{G(e^{i\omega}, \hat{\theta}_N)\}$ of the estimated frequency response, and is plotted as a solid line in Figure 5.1.

In relation to this estimation scenario, a seminal result is that this variability of the frequency response estimate $G(e^{i\omega}, \hat{\theta}_N)$ may be approximated as [162, 164, 167, 168]