Hurwitzin Algebra and its Application to the
FFT Synthesis *

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Abstract

The main idea of the paper is that fast algorithms, like FFT, can be made more efficient in the context of an algebra, rather than in the more singular quaternion or complex algebras structure. However, the complex algebra structure can then be recovered as a projection from the larger algebra in which it is embedded. Namely, the 12-dimensional algebra (hurwitzin algebra) having the basis elements associated with the integer Hurwitz quaternions is introduced. The computational aspects of the hurwitzin arithmetic are considered. The overlapped fast algorithms of two-dimensional discrete Fourier transform of an RGB image are also developed.

Keywords: quaternion algebra, hurwitzin, FFT

1 Introduction

The basis for the well-known “overlapped” one-dimensional FFT [1] is the possibility of obtaining additional computational advantages at the expense of the redundancy of representation of complex basis functions with respect to a real input signal $x(n)$. Putting it more exactly, the possibility of constructing overlapped algorithms exists due to the presence of a non-trivial automorphism (the complex conjugation) of complex field $\mathbb{C}$, acting identically upon $\mathbb{R}$.

Actually, let

$$\hat{x}(m) = \sum_{n=0}^{N-1} x(n) \exp \left\{ 2\pi i \frac{mn}{N} \right\}, \quad m = 0, \ldots, N - 1, \quad N = 2^r, \quad x(n) \in \mathbb{R}. \quad (1)$$

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Let us form an auxiliary sequence
\[ z(n) = x(2n) + i x(2n+1) = x_0(n) + i x_1(n). \] (2)

\[ \hat{z}(m) = \sum_{n=0}^{\frac{N}{2}-1} z(n) \exp\left\{2\pi i \frac{mn}{N}\right\}, \quad m = 0, \ldots, \frac{N}{2} - 1. \]

Then, “partial spectra”
\[ \hat{x}_0(m) = \sum_{n=0}^{\frac{N}{2}-1} x_0(n) \exp\left\{2\pi i \frac{mn}{N}\right\}, \quad \hat{x}_1(m) = \sum_{n=0}^{\frac{N}{2}-1} x_1(n) \exp\left\{2\pi i \frac{mn}{N}\right\} \] (3)
can be found from the relations
\[ 2\hat{x}_0(m) = \hat{z}(m) + \overline{\hat{z}(-m)}, \quad 2i\hat{x}_1(m) = \hat{z}(m) - \overline{\hat{z}(-m)}. \] (4)
The full spectrum reconstruction is realized by relation
\[ \hat{x}(m) = \hat{x}_0(m) + i \hat{x}_1(m) \exp\left\{2\pi i \frac{2m}{N}\right\}. \] (5)

Note that the computer-aided transition to the complex-conjugate number does not result in additional arithmetic operations.

The majority of fast algorithms (of Cooley-Tuckey type) of the discrete Fourier transform (DFT) have the complexity
\[ W(N) = \lambda N \log_2 N + O(N), \]
where the constant \( \lambda \) characterizes a particular scheme of the algorithm [5]. Thus, the complexity of the “overlapped” FFT is
\[ W(N) = \frac{1}{2} \lambda N \log_2 N + O(N). \]

If the technique discussed above is used for multidimensional DFTs (in particular 2D transforms) then certain difficulties may be encountered: the field \( \mathbb{C} \) has “too few” automorphisms admitting repeated overlaps for each argument with the possibility of the subsequent separation of spectra. This reasoning leads to the necessity of immersion of the field \( \mathbb{R} \) into algebraic structures possessing a sufficiently large number of trivially implemented automorphisms over \( \mathbb{R} \).

In some papers (e.g. [7], [8], [6]) I used this approach to solve the problem of DFT fast algorithms synthesis. The idea of this approach is as follows: we explicitly construct the immersion of the complex field into the algebraic structures having sufficient number of trivially implemented automorphisms. Somewhat increased computational complexity of the main arithmetic operations in this case can be compensated by the possibility of the “overlapped” calculation of spectrum portions. This is guaranteed by the existence of the