Completeness Theorems and $\lambda$-Calculus

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Abstract. The purpose of this note is to present a variation of Hindley’s completeness theorem for simply typed $\lambda$-calculus based on Kripke model. This variation was obtained indirectly by simplifying an analysis of a fragment of polymorphic $\lambda$-calculus [2].

1 Introduction

One the most important problem in proof theory is the status of impredicative definitions. Since the sharp criticism of Poincaré [15] one of the goal of Hilbert’s program was precisely to show that such “circular” definitions cannot lead to contradictions. A typical example of impredicative definition is Leibnitz definition of equality, which defines “$a$ is equal to $b$” by the formula

$$\forall X. X(a) \rightarrow X(b) \quad (\ast)$$

Here $X(x)$ ranges over all possible properties. In particular it could be the property $P(x)$

$$P(x) \leftrightarrow_{\text{def}} x \text{ is equal to } b$$

and there is an apparent circularity. If we have a logic with a given equality $=$, it is clear that $(\ast)$ is equivalent to $a = b$: we have indeed that $a = b$ and $\phi(a)$ implies $\phi(b)$, and conversely, if $P(a)$ implies $P(b)$ for any property $P(x)$ we can take $P(x) \leftrightarrow_{\text{def}} a = x$ and we get $a = b$ since $a = a$. In this case, an apparent impredicative definition is equivalent to a predicative one.\footnote{The purpose of the “reducibility axiom” [16] is precisely to postulate that one can always replace an impredicative definition by a predicative one. Theorems 2 and 5 are instances where a priori impredicative definitions can be replaced by predicative ones.}

The intuitions of Poincaré have been confirmed by several works [3], which show that impredicative definitions are proof theoretically very strong. According to Gödel [5], it is precisely the use of impredicative definitions that separates classical mathematics from intuitionistic mathematics, much more than the use of excluded middle (and a similar view is now taken by Martin-Löf).

One breakthrough was accomplished in the 60s by G. Takeuti [18], who showed that the first level of impredicative definitions, so called $\Pi_1^1$ comprehension, can be reduced to a strong form of inductive definitions. G. Takeuti
introduces a stratification of $\Pi^1_1$ comprehension, and the first level, which we shall call \textit{strict} quantification, is obtained by limiting the quantification over predicate $\forall X.\phi(X)$ to formulae $\phi(X)$ which contain only first order quantification. In order to interpret this fragment we need only inductive definitions in a form already considered by Brouwer and thus Takeuti’s result shows that strict $\Pi^1_1$-quantification can be understood intuitionistically. It is quite remarkable that most use of impredicative definitions are done at this level. For instance, Leibnitz equality explained above uses only a strict quantification. Another example is provided by the greatest lower bound of a collection of reals. We represent a real as a Dedekind cut, i.e. a set of rational numbers which is downward closed. If the collection of real numbers is represented by a formula $\phi(X)$, the greatest lower bound, as a set of rationals, is the intersection of all properties satisfying $\phi(X)$. It can thus be represented by the formula $P(q)$ defined by

$$P(q) \leftrightarrow_{de.f} \forall X.\phi(X) \to X(q)$$

Takeuti’s reduction was quite indirect: it was based first on ordinal analysis, and then an intuitionistic proof that the corresponding ordinal system is well-founded. It has been greatly simplified by W. Buchholz [4], by using the $\Omega$-rule. One main intuition can be found in Lorenzen [8]: it is possible to explain the classical truth of a statement $\forall X.\phi(X)$ where $\phi$ does not have any quantification on predicates, by saying that $\phi(X)$ is \textit{provable}, where $X$ is a variable. The key point is that we know how to express classical provability of such formulae using inductive definitions. Indeed the rules of $\omega$-logic provides an intuitionistic way of explaining the truth of arithmetical formulae such as $\phi(X)$, which contains free variables ranging over predicates [13, 9, 10].

For instance, it can be seen in this way that $\forall X.X(5) \to X(5)$ is valid, without having to consider the notion of an arbitrary subset of $\mathbb{N}$, by checking instead that the formula $X(5) \to X(5)$ is provable. This idea of replacing a quantification over an arbitrary subset by a syntactical quantification over a free predicate variable will play an important rôle in this note.

In a previous work [2] we used the idea of the $\Omega$-rule to analyse the system $F_0$, which is a natural restriction of system $F$, with only strict $\Pi^1_1$-quantification. This corresponds closely to the system analysed in [17]. We showed that, for this fragment, normalisation could be proved in Peano arithmetic. We learnt since then that a similar analysis had been done by I. Takeuti [18], following however the method of ordinal analysis of G. Takeuti, and showing that an upper-bound for the restricted system $F$ is $\epsilon_0$. The argument of [2] was simplified by Buchholz. In this version, we have to use a Kripke semantics where worlds are contexts of simply typed $\lambda$-calculus.

\footnote{The corresponding system of inductive definition is called ID$_1$. Stronger forms of inductive definitions are needed to interpret $\Pi^1_1$-comprehension in general, and the intuitionistic status of these stronger forms is not clear.}