On Semi-perfect 1-Factorizations*

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Abstract. The perfect 1-factorization conjecture by A. Kotzig [7] asserts the existence of a 1-factorization of a complete graph $K_{2n}$ in which any two 1-factors induce a Hamiltonian cycle. This conjecture is one of the prominent open problems in graph theory. Apart from its theoretical significance it has a number of applications, particularly in designing topologies for wireless communication. Recently, a weaker version of this conjecture has been proposed in [1] for the case of semi-perfect 1-factorizations. A semi-perfect 1-factorization is a decomposition of a graph $G$ into distinct 1-factors $F_1, \ldots, F_k$ such that $F_1 \cup F_i$ forms a Hamiltonian cycle for any $1 < i \leq k$. We show that complete graphs $K_{2n}$, hypercubes $Q_{2n+1}$ and tori $T_{2n \times 2n}$ admit a semi-perfect 1-factorization.

1 Introduction

In this paper we deal with 1-factorizable graphs, i.e. graphs whose edges can be decomposed into 1-factors (perfect matchings). Clearly, taking the union of any two 1-factors $F_i$ and $F_j$ gives a 2-factor: a spanning subgraph consisting of a set of vertex-disjoint cycles. Additionally, if $F_i \cup F_j$ is connected it forms a Hamiltonian cycle and the corresponding 1-factors $F_i, F_j$ are said to form a perfect pair.

It is widely known that a complete graph $K_{2n}$ is 1-factorizable, see e.g. [8] for a survey. In his 1963 paper [7], A. Kotzig conjectured that for every $n \geq 2$ the complete graph $K_{2n}$ can be decomposed into $n-1$ one-factors in such a way that any two of them form a perfect pair. Despite an extensive effort, this conjecture is still open. Currently it is known that such perfect factorization exists if either $n$ is prime, $2n-1$ is prime or $2n \in \{16, 28, 36, 40, 50, 126, 170, 244, 344, 730, 1332, 1370, 1850, 2198, 3126, 6860\}$ (the references can be found in [10]).

One possible application of the perfect 1-factorization comes from the area of wireless communication. In [3] the problem of building a topology for an ad-hoc network of Bluetooth devices is addressed. As each device can communicate with exactly one other device at a time, the communication pattern at a given time forms a matching. In a bandwidth-efficient topology, a number $k$ of 1-factors is used for communication in a time-multiplexed fashion, where $k$ is

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the parameter of the network: larger values of \( k \) increase the robustness and
decrease the diameter of the network while introducing more communication
overhead due to interference. To achieve fairness, the chosen 1-factors should
cover all edges. Moreover, any two of them should form a connected graph. In
[3], a 1-factorization of \( K_{2n} \) into \( F_1, \ldots, F_{n-1} \) is presented, such that any two
1-factors from \( \{ F_1, \ldots, F_p \} \) form a perfect pair, where \( p \) is the smallest prime
factor of \( 2n - 1 \). Hence, the possible choices of the parameter \( k \) depend on the
number of vertices which is not very convenient.

As a step towards the general solution, [1, 6] propose to solve a weaker version
of the conjecture: find a 1-factorization of a graph \( G \) into 1-factors \( F_1, \ldots, F_k \)
such that \( F_1 \cup F_i \) forms a Hamiltonian cycle for any \( 1 < i \leq k \). Such a 1-
factorization will be called semi-perfect.

The semi-perfect 1-factorization conjecture has application also to the topo-
logical graph theory. In [5], the genus of joins and compositions of graphs is
studied. According to [1]:

This kind of edge coloring (i.e. semi-perfect 1-factorization) of the
cubes (and graphs in general) would lead to an improvement of exist-
ing genus results for joins and compositions of these graphs...

In this paper we show that complete graphs \( K_{2n} \), hypercubes \( Q_{2n+1} \) and tori
\( T_{2n \times 2n} \) admit a semi-perfect 1-factorization.

The potential of semi-perfect 1-factorizations in the topology design is yet to
be investigated.

2 Preliminaries

Unless stated otherwise, we consider simple undirected graphs. A 1-factor (i.e.
perfect matching) of a graph \( G \) is a spanning subgraph in which all vertices have
degree 1. A 1-factorization is a decomposition of the set of edges into distinct
1-factors.

A 1-factorization of a graph \( G \) into 1-factors \( F_1, \ldots, F_k \) is called semi-perfect
if \( F_1 \cup F_i \) forms a Hamiltonian cycle (i.e. connected 2-factor) for any \( 1 < i \leq k \).

Instead of constructing the particular 1-factors \( F_i \), we shall construct the
respective Hamiltonian cycles \( F_1 \cup F_i \) according to the following definition:

**Definition 1.** Let \( G \) be a graph, \( P \) be a 1-factor of \( G \) and \( H_1, \ldots, H_k \) be Hamil-
tonian cycles of \( G \) such that each \( H_i \) contains \( P \) as its subset. Furthermore,
let each edge of \( G \setminus P \) be covered by exactly one cycle \( H_i \). We call the set
\( \{ H_1, \ldots, H_k \} \) a \( P \)-cover of \( G \).

It is easy to see that from given \( G, P \) and \( H_1, \ldots, H_k \) that form a \( P \)-cover of
\( G \) we can construct a semi-perfect 1-factorization of \( G \). Indeed, let \( P_1 := P \) and
\( P_{i+1} := H_i \setminus P \) for each \( 1 \leq i \leq k \). The 1-factors \( P_1, \ldots, P_{k+1} \) form a semi-perfect
1-factorization of \( G \).

In order to construct semi-perfect 1-factorizations of complete graphs and
hypercubes we shall investigate the following two graph operations: