On Wintner’s Conjecture About Central Configurations

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Abstract. According to Wintner, the study of central configurations in celestial mechanics may be reduced to an extremality problem. Wintner’s Conjecture amounts to saying that the corresponding extremal zeroes for each fixed number \( n \) of different masses is finite. By the author’s Finite Kernel Theorem it follows that the corresponding number of extremal values is finite for each fixed \( n \). Thus, Wintner’s Conjecture will be true or false according to whether there will be only a finite number of extremal zeroes or not. This gives thus a new way of attacking Wintner’s Conjecture.

Keywords: Celestial Mechanics, Central Configurations, Wintner’s Conjecture, Extremal Values, Extremal Zeroes, Finite Kernel Theorem.

The determination of central configurations in astromechanics is a fascinating problem since the time of Euler and Lagrange. In fact, Euler had shown that for three arbitrarily given distinct masses, there exist exactly three distinct collinear central configurations, and Lagrange had shown that the equilateral triangle and only the equilateral triangle is a non-collinear central configuration for three arbitrary masses. In the last century Moulton had proved that for arbitrary \( n \) distinct masses the number of collinear central configurations is \( n! \) or \( n!/2 \), cf. [1]. In more recent time, A. Wintner had shown that, for four masses the regular tetrahedron and only the regular tetrahedron is a non-flat central configuration. Wintner had even announced a conjecture to the effect that for arbitrarily fixed number \( n \) of masses, the number of possible central configurations is always finite. This conjecture had resisted all attacks by quite a number of specialists, including e.g. Smale and his followers by some topological method created \textit{ad hoc}, cf. \([5, 6]\) and \([2]\).

The actual determination of central configurations for a fixed \( n \) is not simple. Thus, in the literature there are only sporadic results of not great significance. In the nineties of the last century the present author had formulated a method of the determination of central configurations in reducing it to the solving of a system of polynomial equations under restrictions also in the form of polynomial equations. It is applied to give an alternative proof of theorems of Euler and Lagrange that the central configurations found by them are the only possible
ones for $n = 3$, cf. [9]. In recent years H. Shi and others had applied this method to find various central configurations of special types for $n \geq 4$, cf. their papers, [3, 4, 11].

What is of great significance is this: Wintner in his classic on celestial mechanics, viz. [8], had shown that the central configurations are in correspondence with the extremal zeroes of some extremalization problem of rational function type and hence also of polynomial type. In applying the Finite Kernel Theorem, (cf. [10], Chap. 5, §5) on such extremality problems, we know that the extremal values of such problems are necessarily finite in number. If it can be shown that for each extremal value there can only be associated a finite number of extremal zeroes, then the total number of extremal zeroes of the problem will be finite, which is just the Wintner Conjecture in question. In any way the above gives an alternative method to attack the interesting and difficult conjecture of Wintner.

To begin with, let us first recall the definition of Central Configuration. Thus, let $t$ be the time which is supposed to be fixed, and $m_i$, $(i = 1, \cdots, n)$ be $n$ masses in question. Let us take a barycentric coordinate system with the center of mass of the system $\{m_1, \cdots, m_n\}$ at the origin (see §322 of [8], similar for references below). With respect to such a barycentric coordinate system let $\xi_i = (x_i, y_i, z_i)$ be the barycentric position of $m_i$, $(i = 1, \cdots, n)$. Set

$$
\rho_{jk} = |\xi_j - \xi_k| = [(x_j - x_k)^2 + (y_j - y_k)^2 + (z_j - z_k)^2]^{\frac{1}{2}}. \tag{1}
$$

Then the potential energy of the system is given by

$$
U = \frac{\sum_{1 \leq j < k \leq n} m_j m_k \rho_{jk}}{\rho_{jk}}. \tag{2}
$$

The Newtonian force acting on the mass $m_i$ is then given by

$$
U_{\xi_i} = (U_{x_i}, U_{y_i}, U_{z_i}), \tag{3}
$$
in which $U_{x_i} = \frac{\partial U}{\partial x_i}$, etc.

By Wintner’s definition, the system $\{m_i, \xi_i | i = 1, \cdots, n\}$ is said to form a **Central Configuration** if the force of gravitation acting on each $m_i, \xi_i$ is proportional to both the mass $m_i$ and the barycentric position vector $\xi_i$, i.e.,

$$
U_{\xi_i} = \sigma m_i \xi_i, \quad \text{for} \quad i = 1, \cdots, n, \tag{4}
$$

where $\sigma$ is some scalar independent of $i$.

It is proved by Wintner that

$$
\sigma = -\frac{U}{J}, \tag{5}
$$
in which (§322 bis)

$$
J = \frac{1}{\mu} \Sigma_{1 \leq j < k \leq n} m_j m_k \rho_{jk}^2, \quad \mu = \Sigma_{i=1,\cdots,n} m_i. \tag{6}
$$