Spatial Pythagorean Hodograph Quintics and the Approximation of Pipe Surfaces

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Abstract. As observed by Farouki et al. [9], any set of C^1 space boundary data (two points with associated first derivatives) can be interpolated by a Pythagorean hodograph (PH) curve of degree 5. In general there exists a two dimensional family of interpolants.

In this paper we study the properties of this family in more detail. We introduce a geometrically invariant parameterization of the family of interpolants. This parameterization is used to identify a particular solution, which has the following properties. Firstly, it preserves planarity, i.e., the interpolant to planar data is a planar PH curve. Secondly, it has the best possible approximation order (4). Thirdly, it is symmetric in the sense that the interpolant of the "reversed" set of boundary data is simply the "reversed" original interpolant. These observations lead to a fast and precise algorithm for converting any (possibly piecewise) analytical curve into a piecewise PH curve of degree 5 which is globally C^1.

Finally we exploit the rational frames associated with any space PH curve (the Euler-Rodrigues frame) in order to obtain a simple rational approximation of pipe surfaces with a piecewise analytical spine curve and we analyze its approximation order.

1 Introduction

Pythagorean hodograph (PH) curves (see the survey [11] and the references cited therein), form a remarkable subclass of polynomial parametric curves. They have a piecewise polynomial arc length function and, in the planar case, rational offset curves. These curves provide an elegant solution of various difficult problems occurring in applications, in particular in the context of CNC (computer-numerical-control) machining.

In the planar case, the properties and various constructions of PH curves have been thoroughly studied, e.g., [1, 6, 8, 7, 18, 23]. Due to the constrained nature of PH curves, all constructions – which are linear in the case of polynomial curves – become *nonlinear* in the PH case. Consequently, they may have more than one solution, and the problem of choosing the 'best' solution has to be addressed, e.g. by analyzing the approximation order or using the rotation index [15, 18, 20, 21, 22].

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Spatial PH curves were introduced by Farouki and Sakkalis in 1994 [5], and they have later been characterized using results about Pythagorean quadruples in the ring of polynomials and quaternion calculus [2, 4, 10]. Spatial PH curves can be equipped with rational frames, which were studied in [3, 13, 17].

Various constructions were also given, e.g. a global method for C^2 interpolation of point data by quintic splines has been presented in [12]. Hermite interpolation of G^1 boundary data was addressed in [17], and C^1 Hermite interpolation by PH quintics was discussed in [9]. In the latter case, the authors identify a family of interpolants to any C^1 Hermite data which depends on two free parameters, and a heuristic choice for them is given. Later, this has also been related to helical interpolants [14].

The present paper is devoted to the problem of C^1 Hermite interpolation by spatial PH quintics, and to the approximation of pipe surfaces and sweeping surfaces. We study the family of interpolants and identify the solution which has the best approximation order, preserves planarity, and is symmetric with respect to the reversion $t \mapsto (1-t)$ of the parameter interval $[0, 1]$.

The remainder of the paper is organized as follows. First we recall some basic facts about quaternion algebra and PH curves. The first part of Section 3 summarizes the approach taken in [9] to the problem of C^1 Hermite interpolation by PH quintics. In the second part we introduce a parameterization of the family of interpolants with respect to a standard position. We prove that this parameterization is geometrically invariant and symmetric.

Section 4 provides a qualitative analysis of the solutions. We give an asymptotical analysis, including approximation order, and we identify the parameter values which preserve planarity. Based on these results, we use optimal solution for converting analytical curves into piecewise PH quintic curves and for the approximation of pipe surfaces. Finally we conclude the paper.

2 Preliminaries

In order to make this paper self–contained, we recall some basic facts about quaternions and Pythagorean Hodograph curves.

2.1 Quaternions

Quaternions (see e.g. [19] for an elementary introduction) are elements

$$\mathcal{A} = a + a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \tag{1}$$

of 4–dimensional real linear space $\mathbb{Q}$ with basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$. The space $\mathbb{Q}$ has the structure of a non-commutative field, where the multiplication is defined by the relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1 \tag{2}$$

of the basis elements, which imply

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{3}$$