Look: Simple Stochastic Relations Are Just, Well, Simple

Ernst-Erich Doberkat*

Chair for Software Technology, University of Dortmund
doberkat@acm.org

Abstract. Simple systems cannot decomposed further. Algebraically, simple systems have only isomorphisms as epis. We characterize simple stochastic relations through different forms of bisimulations for the case that the underlying spaces are Polish, and analytic, respectively. This requires a closer investigation of bisimulations, congruences and their mutual relationship. We provide a complete characterization of simple stochastic relations for analytic spaces.

1 Introduction

An algebraic structure which is isomorphic to each of its non-trivial factor spaces is called simple. Take e.g. a simple and non-trivial group $G$ and an epimorphism $\phi : G \to H$, then $\phi$ is an isomorphism, because the factor system $G/\ker(\phi)$ is isomorphic to $G$, thus the kernel $\ker(\phi)$ is trivial, cp. [8, p. 104]. Thus a system $S$ is simple if each epimorphism $S \to T$ is an isomorphism. On the other hand, the very close connection between simple systems and trivial bisimulations is well known in the theory of coalgebras: a system is simple iff it has only trivial bisimulations.

We show in this paper that the investigation of simple stochastic relations through bisimulations is fruitful as well. While in coalgebras heavy use is being made of weak pullbacks, this is not possible for stochastic relations, since they are not available there — in fact, one is glad to have semi-pullbacks [6]. Hence one has to bypass this difficulty at the cost of some rather technical constructions. A further technical point to be considered concerns the structure of the base space. Stochastic relations can be defined on top of arbitrary measurable spaces, but the probabilistic structure of these spaces is too poor to be of much use to us. Thus we resort to a richer structure, viz., Polish spaces and their Borel images, analytic spaces. A closer look will reveal that a careful distinction between these spaces will be required, since stochastic relations on them are different in subtle ways, as we will see.

Bisimulations are usually defined through spans of morphisms, and it turns out that we need to capture different conditions on equivalence relations through

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bisimulations in order to appreciate all properties of simple relations. We introduce 2-bisimulations as those bisimulations that are based on a set-theoretic relation on the space involved, so that the morphisms are just the projections. For our discussion those 2-bisimulations are of interest that are defined through the kernels of morphisms, or, equivalently, through congruences; we call them smooth. There is a very close relationship between congruences and such smooth 2-bisimulations, since we show that each congruence gives rise to such a smooth 2-bisimulation (this is easy in the theory of coalgebras, it turns out to be rather hard work for the stochastic case). This observation yields immediately that a stochastic relation has only trivial congruences iff it has only trivial smooth 2-bisimulations.

This characterization is provided for the case that the spaces on which the stochastic relations are built are Polish. Going a step further to analytic spaces (hence to Borel images of Polish spaces) indicates that we need a further kind of 2-bisimulations that are called weak (they focus on Borel sets that are invariant under the congruence and leave other Borel sets alone). In the Polish case we can show that there is no difference, but in the analytic case this is presumably not the case. We prove that a stochastic relation for relations over Polish spaces is simple iff it has only the trivial congruence iff smooth as well as weak 2-bisimulations are trivial. It is shown that if there can be at most one morphism into a simple stochastic relation, then the relation is simple (the converse holds as well, but under a restrictive condition); Figure 1 gives an overview. This is essentially the situation for the analytic case, too, but the equivalence of weak and smooth 2-bisimulations is a bit weaker; Figure 2 provides a pictorial summary for this case as well. All this leads to a complete characterization of simple analytic relations by injective measurable maps into the unit interval of the reals. It implies that final systems do not exist unless the system is truly probabilistic: then there is exactly one.

Final coalgebras are used by Rutten [10, 11] for establishing a calculus of coinduction. Since the structure of simple systems is much poorer for stochastic relations, such an endeavor cannot be expected to be as fruitful as in the general coalgebraic case, but we indicate that the identification of simple relations may occasionally be helpful nevertheless. We derive in the full paper [5] an explicit representation of the number of heaps that is central to the analysis of Williams’

**Fig. 1.** Simple systems: the Polish case