3 Stationary Processes

3.1 Introduction

Much of the fundamental work done in the area of reduced order dynamic system optimization [1, 2] has been done in a setting which allows for a solution involving only algebraic equations rather than differential equations. This setting is in terms of time invariant linear systems driven by white noise processes. The basic assumption is that the processes considered are in steady state in a statistical sense, with a bounded constant second moment matrix. Technically, we refer to such systems as stationary stochastic processes [3]. The processes are dynamic in the sense that their states are moving with time. However, the statistics are constant. The advantages of considering such processes are

1. Only basic calculus is needed to find the necessary conditions for an optimal solution.
2. The solution is easy to implement.
3. In practice, considering systems after the transients due to initial conditions have died out is often the only practical option.

The negative aspects of the stationary setting include the facts that we would like to consider unstable processes, as well as time variable systems, and systems for which transient effects are important. Nevertheless, it is worth considering stationary processes separately, since they do form an important class of problem.

3.2 Unbiased Linear Filters

We shall consider systems described by the dynamical equations

$$\dot{\mathrm{x}} = \mathrm{Ax} + \mathrm{Bu} + \theta w \tag{3.2.1}$$

A.A. Jalali, C.S. Sims, and P. Famouri: Reduced Order Systems, LNCIS 343, pp. 27–57, 2006.

where A, B and θ are known constant matrices, x is the n dimension state vector, u is the m dimensional control vector, and w is a p dimensional of zero mean white noise with covariance matrix

$$E\left\{w(t)w^T(\tau)\right\} = \hat{Q}\delta(t-\tau). \tag{3.2.2}$$

The output equation is

$$y = Cx \tag{3.2.3}$$

and the measurement is

$$m = Cx + v = y + v \tag{3.2.4}$$

where C is a known constant matrix, and where v is a k dimensional vector of zero mean white noise with covariance matrix, $\hat{R}$, i.e.

$$E\left\{v(t)v^T(\tau)\right\} = \hat{R}\delta(t-\tau).$$

There are many problems one can consider in this general setting. We will first consider an estimation problem where B is zero. The problem under consideration is to estimate $z(t) = Lx(t)$ with a filter of the form

$$\dot{\hat{Z}} = F\hat{z} + Km. \tag{3.2.5}$$

When $L = I$, this problem has a well known solution when the performance measure to be minimized is the mean squared error

$$J = \lim_{t\to\infty} E\left\{e_f^T(t)e_f(t)\right\} \tag{3.2.6}$$

where $e_f = x - \hat{z} \equiv x - \hat{x}$. The solution is generally referred to as the steady state Kalman filter or the Weiner filter. With $\hat{z} = \hat{x}$, the solution is of the form

$$\dot{\hat{z}} = A\hat{z} + K[m - C\hat{z}] = \dot{\hat{x}} = A\hat{x} + K[m - C\hat{x}] \tag{3.2.7}$$

where

$$K = P_f C^T \hat{R}^{-1} \tag{3.2.8}$$

And P_f is the positive semi-definite solution to the algebraic matrix Riccati equation

$$0 = AP_f + P_f A^T - P_f C^T \hat{R}^{-1} C P_f + \theta\hat{Q}\theta^T. \tag{3.2.9}$$