4 Estimation over Finite Time Interval

4.1 Introduction

In this chapter we shall consider the estimation problem over a finite interval of time. In formulating such problems we limit the allowable structure of the estimator so that the number of computations is kept within reason. In solving the problems presented here we primarily use the matrix version of the minimum principle [1] as our method of derivations. In certain cases it is preferrable to solve the problem using the innovations method [2, 3] and the orthogonal projection principle [4], so we shall introduce that principle as well, and it will be used extensively in future chapters. The importance of the material presented here is that we can apply our methods to non-stationary stochastic processes. This is the advance that Kalman Filtering made over Wiener filtering [6], only we are doing it in a reduced order state space setting. The systems considered can be time variable, as when one linearizes equations about a nominal trajectory which varies with time. Alternatively, we may look at stable systems during the time interval for which their initial conditions are having significant impact on the response. Or we may consider unstable systems for which stationary conditions are never met. Thus this chapter opens up many new possibilities, although we must still restrict ourselves to linear systems, described by state space equations.

4.2 Optimal Unbiased Estimators of Fixed Order

The dynamical model considered in this section is of the form

$$\dot{x}(t) = A(t)x(t) + \theta(t)w(t) \tag{4.2.1}$$

and the observation model is of the form

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$$m(t) = C(t)x(t) + v(t) . \tag{4.2.2}$$

The process noise, $w(t)$, and the measurement noise $v(t)$, are independent zero mean white processes with covariance matrices $\hat{Q}(t)$ and $\hat{R}(t)$ respectively. That is

$$E\left\{w(t)w^T(\tau)\right\} = \hat{Q}(t)\delta(t-\tau)$$

$$E\left\{v(t)v^T(\tau)\right\} = \hat{Q}(t)\delta(t-\tau) \tag{4.2.3}$$

where these must be positive semi-definite. The noise is independent of the initial statistics $E\left\{x(t_0)\right\} = \mu_x(t_0)$; $E\left\{x(t_0)x^T(t_0)\right\} = P_{xx}(t_0)$. Our problem is to design a filter to estimate $z(t) = L(t)x(t)$, where we are constrained to use a filter of the form

$$\dot{\hat{z}}(t) = F(t)\hat{z}(t) + K(t)m(t) . \tag{4.2.4}$$

The problem of interest is to select $F(t)$ and $K(t)$, and $\hat{z}(t_0)$, to minimize the error criterion

$$J = E\left\{\int_{t_0}^{t_f} e^T(t)Ue(t)\,dt + e^T(t_f)se(t_f)\right\} \tag{4.2.5}$$

where U and S are positive semi-definite and positive definite weighting matrices respectively whose precise values are not important in the final solution. It is also required that the estimate be unbiased, that is

$$E\left\{e(t)\right\} = E\left\{z(t) - z(t)\right\} = 0 \,\forall\, t \in \left[t_0, t_f\right] . \tag{4.2.6}$$

This problem has been treated in [7]. The differential equation for the error is

$$\dot{e} = \dot{z} - \dot{\hat{z}} = \dot{L}x + L\dot{x} - \dot{\hat{z}}$$

or

$$\dot{e} = \left(\dot{L} + LA\right)x + L\theta w - F\hat{z} - K\left(Cx + v\right) \tag{4.2.7}$$

where the reader should note that these matrices are time variable even though we have dropped the explicit notation indicating this. If we substitute $\hat{z} = Lx - e$ in (4.2.7), we obtain the expression

$$\dot{e} = \left(L + LA - KC - FL\right)\hat{x} + Fe + L\theta w - Kv . \tag{4.2.8}$$

The error equation can be decoupled from the state equation if we constrain the matrices F and K to algebraically related as