6 Stochastic Control over Finite Time Intervals

6.1 Introduction

Here we shall discuss finite time interval problems where the main idea is to control a system, and where estimation, if it is a part of the problem at all, is of secondary importance. We shall discuss problems with and without dynamic compensators. Dynamic compensators are important for the case when full state measurements are not available, so that an observer is useful [1]. They are also important for the case where only a noisy measurement of the observation is available and filtering of the noise is necessary. Such problems have been considered in [2, 3]. We will see that only under very special circumstances do such problems have elegant solutions. Such is the case when full order compensators are used, and the result known as the separation theorem [4], is probably the most elegant result in all of systems theory. We will begin our examination of Stochastic Control problems by examining the output feedback control problem which has been studied by Axsater [5].

6.2 The Basic Stochastic Control Problem

The system under consideration is described by the dynamical equation

$$\dot{x}(t) = A(t)\,x(t) + B(t)u(t) + \theta(t)w(t)\,. \tag{6.2.1}$$

It is easily recognizable as the Kalman filter when a control input is available. Notice that the gain $\hat{K}(t)$ does not depend on the weighting matrices for the control problem, and may be evaluated as if this were a standard Kalman filtering problem. Furthermore, the control gain, $K(t)$, is computed exactly as one would for the case when the complete vector $x(t)$ is available, i.e. as in section (6.2) for the case when $C = I$. This separate

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calculation of the gains for filtering and control is a well known result in stochastic control theory [4]. However, our presentation is quite different, and requires no prior assumption of a gaussian process. For any white noise situation the result is the best linear result. The separation can be interpreted as the separating of a single two-point boundary value problem into an initial value problem and a separate terminal value problem. An important point is that this situation is not unique, but that there are many solutions corresponding to different values of $F(t)$. However, the separation solution is certainly the most desirable from the view point of simplicity of the design equations. We must recall, however, that this solution requires that the compensator be of order n, so that the filter is a full order Kalman filter, and not a reduced order filter.

It is interesting to note that we have obtained an optimal estimate even though estimation was not a part of the performance criterion. The estimate and the error are orthogonal, i.e.

$$E\left\{\left[x(t)-\hat{z}(t)\right]\hat{z}^T(t)\right\}=0$$

which is simply another way of writing down the fact that $\Omega_1(t)$ is zero when F is chosen according to (6.3.16). The other solutions to the problem do not have this nice property.

In fact, this orthogonality principle is a powerful tool which we will develop further in later chapters where $w(t)$ is zero mean white noise with covariance matrix, $\hat{Q}(t)$ and the available observations are

$$y(t)=C(t)x(t)\,. \tag{6.2.2}$$

The control is assumed to be of the linear form

$$u(t)=K(t)y(t) \tag{6.2.3}$$

where $K(t)$ is chosen to minimize the quadratic performance measure

$$J=E\left\{\int_{t_0}^{t_f}\left[x^T(t)Q(t)x(t)+u^T(t)R(t)u(t)\right]dt+x^T(t_f)Sx(t_f)\right\}. \tag{6.2.4}$$

If $C(t)$ is an invertible matrix, then this problem has a very nice solution; otherwise a matrix non linear two-point boundary value problem results as was first pointed out by Axsater [5].

The performance measure may be written down in terms of the second moment of the state vector

$$J=\mathrm{tr}\left\{\int_{t_0}^{t_f}\left[Q(t)P_{xx}(t)+C^T(t)K^T(t)R(t)K(t)C(t)P_{xx}(t)\right]dt+SP_{xx}(t_f)\right\} \tag{6.2.5}$$