# 7 Innovation Approach to Reduced Order Estimation

## 7.1 Introduction

In this chapter we will rederive some of the results that have been previously obtained in Chapter three using a concept of "reduced order innovation process." The concept of "reduced order innovation process" we feel captures the qualitative essence of how useful information is extracted from the given measurements when a reduced order estimator is used. For example, consider an $n^{th}$ order linear dynamical system for which one has built an $\ell^{th}$ order (with $\ell < n$) filter to estimate $z(t) = L(t)x(t)$ where $x(t) \in R^n$ is the state of stochastic system at time $t$ and $L(t) \in R^{\ell \times n}$. If one implements such a reduced order filter, then at any given time t, one would only have an estimate of $z(t) = L(t)x(t)$, unlike the Kalman filter where one would have an estimate of the entire state of the system $x(t)$. The concept of "reduced order innovation process" deals with how one extracts the "useful" information from the new measurements based only on the estimate of $z(t)$.

This chapter is organized as follows. In section 7.1 we re-analyze a static problem similar to the one studied in Chapter 2, but using a new approach that is central to our concept of reduced order innovation process. In section 7.2 we define the reduced order innovation process for a linear system driven by white noise and derive the "reduced order Wiener-Hopf equation" which is a generalization of the well known Wiener-Hopf equation. In sections (7.4) and (7.5) we obtain formula for optimal reduced order filter and smoother respectively using the innovations approach and the "reduced order Wiener-Hopf equation." Much of the material in this chapter is based on Nagpal et.al. (1989 a and b) where analogous results for discrete time system are also presented.

A.A. Jalali, C.S. Sims, and P. Famouri: Reduced Order Systems, LNCIS 343, pp. 109–135, 2006.

## 7.2 A Static Problem

In this section we consider a static problem similar to the one in the section 2.2, as a means of developing the notion of "reduced order innovation process."

First, we briefly introduce the notation. Let H be the Hilbert space of real, scalar and zero-mean random variables, with inner product for two random variables $u$ and $v$ defined as

$$u, v \in H, \quad < u, v >= E\{u\, v\} \tag{7.2.1}$$

where $E\{.\}$ denotes the expected value operator. Two vectors $x$ and $y$, all elements of which are in $H$, are said to be orthogonal to each other if $E\{x_1 y_1\} = 0$ for all elements $x_i$ of $x$ and $y_i$ of $y$. This can be written in a more compact form as

$$x \perp y, \text{ iff } E\{x^T y\} = 0\,. \tag{7.2.2}$$

It is well known that estimate of $x$, denoted by $\hat{x}$, is the best minimum variance linear estimate of $x$ given some observations y if and only if: $(x - \hat{x}) \perp y$.

This is the general form of Wiener-Hopf equation. Let us now move to the static problem we wish to consider in this section.

### The Static Problem:

Let $x \in R^n$, $y \in R^m$, $z \in R^\ell$, $v \in R^m$, $s \in R^p$; $p$, $m \leq n$, $\ell < n$, $S \in R^{p \times n}$, $C \in R^{m \times n}$, $L \in R^{\ell \times n}$; all are full row rank matrices with $S^T \in \mathrm{im}\left[L^T C^T\right]$.
$S^T \in \mathrm{im}\left[L^T C^T\right]$ means that columns of $S^T$ are spanned by columns of $L^T$, $C^T$ and

$$H = \begin{bmatrix} x \\ v \end{bmatrix}.$$

Given a priori data: $z = Lx$, $E\{z\} = \mu$, $\mathrm{Var}\{z\} = C_{zz}$, $E\{v\} = 0$ and $E\{x\, v^T\} = 0$, $\mathrm{Var}\{v\} = C_{vv\lambda}$.

The measurement: $y = Cx + v$.

The problem: Obtain the unbiased minimum variance linear estimate of $s = Sx$ where $S$ is a given full row rank matrix of dimension $p \times n$. An estimate $\hat{s}$ of s is called unbiased if $E\{\hat{s}\} = E\{s\}$.