# 8 Linear Two-Point Boundary Value Processes

## 8.1 Introduction

The construction of estimators for linear two-point boundary value processes (TPBVP's) has been receiving greater attention in recent years [1-4]. This interest reflects a desire to estimate processes governed by ordinary differential equations with constraints at each end point of an interval. TPBVP's occur frequently in physics and engineering. For example, partial differential equations in temporal steady-state often lead to TPBVP's.

The estimation problem for TPBVP's is complicated by their acausal nature. This noncausality implies that a TPBVP is a non-Markov process and, hence, the filtering problem can not be formulated for the non-Markov TPBVP. Since smoothing is inherently an acausal process, the smoothing problem can be formulated for a TPBVP. However, the solution of the smoothing problem can not be based upon any of the classical smoothing algorithms that use a Kalman filter. The smoothing problem for TPBVP's has been solved by the method of complementary models for minimum-variance linear estimation [1-3]. For an $n^{th}$ order TPBVP the smoother is given by a $2n^{th}$ order two-point boundary value system [1, 3].

Although the TPBVP is not a Markov process it is possible to obtain a Markov model of the TPBVP by embedding its state vector in a larger state vector [1]. It is possible to apply the Kalman filter and the classical smoothing algorithms to the Markov model of the TPBVP. In general, the order of the Markov model is twice as large as the non-Markov model. This implies that the order of the Kalman filter is of the same order as the non-Markov smoother, and the order of the Markov model smoother is twice as large. There are special cases where the order of the Markov model is less than twice as large as the non-Markov model and even some cases where they have the same order (called a separable TPBVP). But in

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general the order of the Markov model is much larger than the order of the non-Markov model. For these reasons the Markov model approach to TPBVP's is not very satisfying and it will not be pursed here. It should be noted that reduced- order estimators for the Markov model of the TPBVP have been constructed using a reduced order complementary model [5].

We shall consider the reduced order smoothing problem for the non-Markov TPBVP. Rather than use a complementary model, which requires concepts from Hilbert space theory and operator theory, we will use the least-squares approach to smoothing. This approach views smoothing as a deterministic least-squares fit problem and it has been applied to the smoothing problem for initial value processes [6, 7, 8]. It only requires well-known techniques from the calculus of variations and it is mathematically simpler than the complementary model approach.

## 8.2 Problem Statement

The $n \times 1$ state vector $x$ for the TPBVP is generated on the interval [0,T] by the linear differential equation

$$\dot{x}(t) = A(t)x(t) + \theta(t)w(t) \tag{8.2.1}$$

with the two-point boundary condition

$$b = V_0 x(0) + V_T x(T) = V x_b . \tag{8.2.2}$$

Although $t$ is used to denote the independent variable it should not be assumed that $t$ represents time; for many TPBVP's the independent variable is a spatial parameter. The plant noise $w$ is a $P \times 1$ vector with the prior Gaussian statistics

$$E\{w(t)\} = 0; \ E\{w(t)w^T(\tau)\} = Q(t)\delta(t-\tau) , \tag{8.2.3}$$

and the boundary condition b is an $n \times 1$ vector with the prior Gaussian statistics

$$E\{b\} = 0; \ E\{bb^T\} = \Pi . \tag{8.2.4}$$

Also, $w$ and $b$ are uncorrelated, and $Q$ and $\Pi$ are positive-definite matrices. In (8.2.2) $x_b$ is the $2n \times 1$ boundary process defined by

$$x_b = \begin{bmatrix} x(0) \\ x(T) \end{bmatrix} \tag{8.2.5}$$