Reduction Strategies for Left-Linear Term Rewriting Systems*

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Abstract. Huet and Lévy (1979) showed that needed reduction is a normalizing strategy for orthogonal (i.e., left-linear and non-overlapping) term rewriting systems. In order to obtain a decidable needed reduction strategy, they proposed the notion of strongly sequential approximation. Extending their seminal work, several better decidable approximations of left-linear term rewriting systems, for example, NV approximation, shallow approximation, growing approximation, etc., have been investigated in the literature. In all of these works, orthogonality is required to guarantee approximated decidable needed reductions are actually normalizing strategies. This paper extends these decidable normalizing strategies to left-linear overlapping term rewriting systems. The key idea is the balanced weak Church-Rosser property. We prove that approximated external reduction is a computable normalizing strategy for the class of left-linear term rewriting systems in which every critical pair can be joined with root balanced reductions. This class includes all weakly orthogonal left-normal systems, for example, combinatory logic CL with the overlapping rules $pred \cdot (succ \cdot x) \to x$ and $succ \cdot (pred \cdot x) \to x$, for which leftmost-outermost reduction is a computable normalizing strategy.

1 Introduction

Normalizing reduction strategies of reduction systems, such as leftmost-outermost evaluation of lambda calculus [2, 11], combinatory logic [7, 11], ordinal recursive program schemata [25] and left-normal term rewriting systems [8, 17, 22] guarantee a *safe* evaluation which reduces a given expression to its normal form whenever it exists. Hence, normalizing reduction strategies play an important role in the implementation of functional programming languages based on reduction systems.

Strong sequentiality formalized by Huet and Lévy [8] is a well-known practical criterion guaranteeing an efficiently computable normalizing reduction strategy for orthogonal (i.e., left-linear and non-overlapping) term rewriting systems. They showed that for every strongly sequential orthogonal term rewriting system R, strongly needed reduction is a computable normalizing strategy, that is,

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by rewriting a redex called a *strongly needed redex* at each step, every reduction starting with a term having a normal form eventually terminates at the normal from. Here, the strongly needed redex is defined as a *needed redex* concerning an approximation of R which is obtained by analyzing the left-hand sides only of the rewrite rules of R. Moreover, Huet and Lévy [8] proved the decidability of strong sequentiality. A simpler proof by Klop and Middeldorp can be found in [12] and a proof based on second order monadic logic and tree automata by Comon in [3].

Inspired by the seminal work by Huet and Lévy [8], several better decidable approximations of left-linear term rewriting systems, for example, NV approximation [21], shallow approximation [3], growing approximation [9, 15], etc., have been investigated in the literature. Moreover, Durand and Middeldorp [6] presented a simple uniform framework for normalizing reduction strategies based on decidable approximations. In all of these works [6, 9, 10, 15], however, the non-overlapping restriction is still required to guarantee that approximated decidable needed reductions are actually normalizing strategies; hence, they cannot be applied to term rewriting systems with overlapping rules such as

$$\begin{cases} \mathit{pred}(\mathit{succ}(x)) \to x \\ \mathit{succ}(\mathit{pred}(x)) \to x. \end{cases}$$

Though it is known [6, 9, 10, 15] that only the left-linearity restriction is necessary for considering decidability issues, the question whether there exists an approximated decidable normalizing strategy for left-linear overlapping term rewriting systems has received quite a bit of attention.

The main purpose of this paper develops decidable normalizing reduction strategies for left-linear overlapping term rewriting systems. The notion of sequentiality defined by Huet and Lévy [8] is naturally adapted to that of externality. An external term rewriting system R guarantees that every reducible term contains an outer needed redex, called an external redex, which remains at an outer position until it is rewritten. Under this new framework, we show that external reduction is normalizing for the class of external *root balanced joinable* term rewriting systems. A root balanced joinable term rewriting system is defined as a term rewriting system in which every critical pair can be joined with *root balanced reductions*. We also show that for weakly orthogonal left-normal systems, the leftmost-outermost reduction strategy is normalizing. For example, the leftmost-outermost reduction strategy is normalizing for combinatory logic CL $\cup \{\mathit{pred}\cdot(\mathit{succ}\cdot x) \to x,\ \mathit{succ}\cdot(\mathit{pred}\cdot x) \to x\}$. Here, combinatory logic CL [2, 7, 11] is the orthogonal term rewriting system having the following rewrite rules:

$$\mathrm{CL} \quad \begin{cases} ((S \cdot x) \cdot y) \cdot z \to (x \cdot z) \cdot (y \cdot z) \\ (K \cdot x) \cdot y \to x. \end{cases}$$

Moreover, our result can be applied to term rewriting systems not having the Church-Rosser property too. For example, the leftmost-outermost reduction strategy is again normalizing for CL $\cup$