On the Undecidability of Coherent Logic

Marc Bezem

Department of Computer Science, University of Bergen,
P.O. Box 7800, N-5020 Bergen, Norway
bezem@ii.uib.no

Abstract. Through a reduction of the halting problem for register machines we prove that it is undecidable whether or not a coherent formula is a logical consequence of a coherent theory. We include a simple completeness proof for coherent logic. Although not published in the present form, these results seem to be folklore. Therefore we do not claim originality. Given the undecidability of the halting problem for register machines the presentation is self-contained.

1 Introduction

As far as we know, Skolem [12] was the first who used coherent logic (*avant la lettre*) to solve a decision problem in lattice theory and to prove the independence of Desargues' Axiom from the other axioms of projective plane geometry. Modern coherent logic, also called finitary geometric logic or even simply geometric logic, arose in algebraic geometry, see for example [5–Sect. 16.4], and is actually a fragment of higher-order logic. In this note we define coherent logic (abbreviated by CL) as the fragment of first-order logic (FOL) consisting of implicitly universally quantified implications of the following form:

$$A_1 \wedge \cdots \wedge A_n \to E_1 \vee \cdots \vee E_m$$

Here the A_i are first-order atoms. In contrast to resolution logic [9], where the E_j must also be atoms, they may here be existentially quantified conjunctions of atoms. Thus the general format of a *coherent formula* reads:

$$A_1 \wedge \cdots \wedge A_n \to \exists \boldsymbol{x}_1.C_1 \vee \cdots \vee \exists \boldsymbol{x}_m.C_m \tag{1}$$

where the C_j are conjunctions of atoms. The special cases $n = 0$, $m = 0$ and no existential quantification, in all possible combinations, are understood to be included. (If the premiss is empty we leave out the $\to$ as well, an empty conclusion is denoted by $\perp$, *falsum*.) A *coherent theory* is a set of coherent formulas. Closed atoms will also be called *facts*.

The fact that first-order logic is semidecidable certainly constitutes an upper bound for the coherent fragment as well. Resolution logic with only constants is decidable, since quantification over finite Herbrand domains can be reduced to propositional logic. Horn clause logic [6] is the format (1) with $m \leq 1$ and E_1 atomic. In the presence of one constant and one unary function symbol, Horn

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clause logic is undecidable [11]. This provides the clue for the undecidability of CL without function symbols, since a unary function $f(x)$ can be replaced by a binary predicate $F(x,y)$ plus coherent axioms $\exists y.F(x,y)$ and $F(x,y) \wedge F(x,z) \to E(y,z)$ as well as congruence axioms for E, which are also coherent. Hence the undecidability result in itself is not surprising, but we show that one can do without all axioms in which E occurs. One can even do away with the constant by using an extra unary predicate. Undecidability of CL can be obtained in many other ways, for example, as an immediate corollary of the linear translation of FOL to CL given in [1]. The current exposition offers an insightful correspondence between computations and proofs.

There are several reasons why coherent/geometric logic is interesting. See [3] for the relevance to computer science. Reasoning in CL is constructive and can be used for, e.g., the constructivization of classical abstract algebra, see [4]. A substantial number of reasoning problems (e.g., in confluence theory, lattice theory and projective geometry) can be formulated *directly* in CL without any clausification or skolemization. This gives some additional benefits in terms of guiding an automated theorem prover and using the proof objects in other logical frameworks. The automation of CL has been studied in [1], inspired by the system SATCHMO [7] for resolution logic.

2 Proof System

CL has a natural proof system which is based on forward ground reasoning with case distinction. Existential quantifiers are eliminated by introducing witnesses. A *witness* is a new constant witnessing the truth of an existential statement. Witnesses play a similar role as eigenvariables in systems of natural deduction and should be chosen completely fresh, not introduced earlier in the proof, not occurring in the theory, nor in the formula to be proven.

In order to elaborate the proof system a bit more, let T be a coherent theory. Assume we have a set I of witnesses and initial constants. The latter constants are the constants occurring in T and in the goal G, the formula to be proven. A *goal* is a closed formula of the same form as a conclusion in a coherent formula. We first explain how to prove a goal and then generalize this to arbitrary coherent formulas. Let X be a set of facts in which only constants from I occur. Together I and X form a so-called (reasoning) *state*. A conjunction of facts is true in this state if all these facts occur in X. A closed formula of the form $\exists \boldsymbol{x}.C$, with C a conjunction of atoms, is true in this state if there exist witnesses $\boldsymbol{w} \in I$ such that $C[\boldsymbol{x}{:=}\boldsymbol{w}]$ is true in the state. A goal is true in a state if at least one of its disjuncts is true in that state. A reasoning *step* in the state (I,X) consists in picking a closed I-instance $C \to D$ of an axiom from T that is invalid in the state. This means that the premiss C is true in the state, but the conclusion D is not.

As an example, consider a state with $I = \{0\}$, $X = \{Nat(0)\}$ and an axiom

$$Nat(x) \to \exists y.(Nat(y) \wedge S(x,y)) \tag{2}$$

Assume we would like to prove $G = \exists xy.(S(0,x) \wedge S(x,y))$. The instance of (2) with $x{:=}0$ is invalid in the current state, since the premiss is true but the conclu-