Löb's Logic Meets the μ-Calculus

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This paper is dedicated to Jan Willem Klop on the occasion of his 60th birthday. Jan Willem's way of doing research is a perfect illustration of the saying Vakmanschap is Meesterschap*.*

Abstract. In this paper, we prove that Löb's Logic is a retract of the modal μ-calculus in a suitable category of interpretations. We show that various salient properties like decidability and uniform interpolation are preserved over retractions. We prove a generalization of the de Jongh-Sambin theorem.

1 Introduction

Fixed points are a central subject of metamathematics. Two flavours are of interest to us here: fixed points of operators that are in some sense guarded and fixed points of monotonic operators. We will study such fixed points in the context of propositional modal logic. The simplicity of propositional modal logic helps us to gain control and overview.

There are two prominent modal logics of fixed points. One is Löb's Logic, aka GL. Löb's Logic is a logic of *guarded fixed points*. It is an important tool in the study of arithmetical self-reference. Löb's Logic has been extensively studied. See the expository papers and books [1], [2], [3], [4], [5].

The other prominent logic is the modal μ-calculus. This logic is a logic for *minimal and maximal fixed points of monotonic operators*. It was designed for applications in Computer Science. This logic was introduced in [6]. See also the survey paper [7].

Both logics are very beautiful and have many desirable properties such as decidability and uniform interpolation.

Johan van Benthem, in his paper [8], showed that Löb's Logic can be faithfully interpreted in the μ-calculus. Moreover, he proved that Löb's Logic has definable fixed points for operators defined by formulas in which the designated variable p occurs only positively. Thus, the μ-calculus can be interpreted in Löb's Logic. Our paper is a commentary on, and an extension of van Benthem's paper. We describe more fully the relationship between both logics: Löb's Logic is a retract of the μ-calculus in a suitable category of interpretations. From this, it follows that properties like decidability and uniform interpolation can be transferred from the μ-calculus to Löb's Logic[1].

[1] Of course, these facts were known already for Löb's Logic. But at least they do receive markedly different proofs via our results.

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Along a different line, van Benthem's arguments are semantical in nature. It is always satisfactory to see semantical arguments replaced by syntactical ones. In this paper we work entirely with syntactical arguments, which are often very simple.

In Section 4, we do a bit more than needed for the rest of the paper. We prove the appropriate generalization of both the de Jongh-Sambin fixed point theorem and van Benthem's theorem that GL has definable minimal fixed points.

1.1 Löb's Logic

Löb's Logic is the logic $\mathsf{K4}$ plus Löb's principle.

LP $\quad \vdash \Box(\Box\phi \to \phi) \to \Box\phi.$

This logic is the logic of upwards well-founded transitive frames.

An occurrence of a variable p in ϕ is *modalized* or *boxed* iff it is in the scope of a necessity operator. Consider a formula $\phi p \boldsymbol{q}$ in which all occurrences of p are boxed. We assume all variables of ϕ are among $p, \boldsymbol{q}$. The de Jongh-Sambin fixed point theorem tells us that there is a formula $\psi\boldsymbol{q}$, with only variables among $\boldsymbol{q}$, such that $\mathsf{GL} \vdash \psi\boldsymbol{q} \leftrightarrow \phi(\psi\boldsymbol{q})\boldsymbol{q}$. Moreover, by the Bernardi-de Jongh-Sambin uniqueness theorem this ψ is unique modulo provable equivalence.

For more information see [1], [2], [3], [4], [5].

1.2 The μ-Calculus

The modal μ-calculus was introduced in [6]. See also the survey paper [7].

For our purposes, the language of the μ-calculus will be the uni-modal language extended with the variable binding operator μp[2]. The formation of $\mu p \cdot \phi$ is allowed precisely if all occurrences of p in ϕ are positive. The μ-calculus is axiomatized by the axioms and rules of K plus the following principle and rule, for ϕ in which all occurrences of p are positive.

$\mathsf{min1}$ $\quad \vdash \mu p \cdot \phi p \leftrightarrow \phi(\mu p \cdot \phi p).$
$\mathsf{min2}$ $\quad \vdash \phi\alpha \to \alpha \;\Rightarrow\; \vdash \mu p \cdot \phi p \to \alpha.$

Frames for the μ-calculus are the usual frames for uni-modal logic. The semantics of $\mu p{\cdot}\phi p$ is as follows. It gives us the minimal fixed point of the operator naturally associated with the formula ϕp and the designated variable p.

We can define a maximal fixed point operator as follows:

$$\nu p \cdot \phi p := \neg\, \mu p \cdot \neg\, \phi \neg\, p.$$

We can easily verify that ν satisfies the following principle and rule.

$\mathsf{max1}$ $\quad \vdash \nu p \cdot \phi p \leftrightarrow \phi(\nu p \cdot \phi p).$
$\mathsf{max2}$ $\quad \vdash \alpha \to \phi\alpha \;\Rightarrow\; \vdash \alpha \to \nu p \cdot \phi p.$

We will write '$\mu \vdash \phi$' for: ϕ is derivable in the μ-calculus.

[2] Usually, the μ-calculus is formulated for a multi-modal language. However, for our present purposes, the extra modalities would do no work.