Böhm's Theorem, Church's Delta, Numeral Systems, and Ershov Morphisms

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Abstract. In this note we work with untyped lambda terms under β-conversion and consider the possibility of extending Böhm's theorem to infinite RE (recursively enumerable) sets. Böhm's theorem fails in general for such sets $\mathcal{V}$ even if it holds for all finite subsets of it. It turns out that generalizing Böhm's theorem to infinite sets involves three other superficially unrelated notions; namely, Church's delta, numeral systems, and Ershov morphisms. Our principal result is that Böhm's theorem holds for an infinite RE set $\mathcal{V}$ closed under beta conversion iff $\mathcal{V}$ can be endowed with the structure of a numeral system with predecessor iff there is a Church delta (conditional) for $\mathcal{V}$ iff every Ershov morphism with domain $\mathcal{V}$ can be represented by a lambda term.

1 Introduction

We suppose the reader knows some lambda calculus, as e.g. in [1], Chapters 6, 7, 8 and 10.

Definition 1.1. (i) *The set of untyped closed lambda terms is denoted by $\Lambda^{\emptyset}$. A* combinator *is an element of $\Lambda^{\emptyset}$.*

(ii) *We denote congruence under beta conversion by $=$.*

(iii) *We write $:=$ for "equal by definition".*

(iv) *We define the following combinators.*

$$
\begin{aligned}
\mathbf{c}_n &:= \lambda f x. f^n x, && \textit{the } \text{Church numerals.}\\
\mathsf{U}^n_k &:= \lambda x_1 \ldots x_n . x_k, && \textit{for } 1 \le k \le n, \textit{ the } \text{projections.}\\
\Omega &:= (\lambda x.xx)(\lambda x.xx).
\end{aligned}
$$

(v) *For lambda terms $\boldsymbol{P} = P_1, \ldots, P_n$ we write*

$$\langle P_1, \ldots, P_n \rangle := \lambda z. z P_1 \ldots P_n.$$

Note that

$$\langle P_1, \ldots, P_n \rangle = \langle Q_1, \ldots, Q_n \rangle \;\Leftrightarrow\; P_1 = Q_1 \;\&\; \ldots \;\&\; P_n = Q_n.$$

The classical theorem of Böhm states the following.

A. Middeldorp et al. (Eds.): Processes... (Klop Festschrift), LNCS 3838, pp. 40–54, 2005.

Theorem 1.2 ([3]). *For all combinators M_1 and M_2 having a β-nf (normal form) the following are equivalent.*

(i) *For all combinators N_1, N_2 there exist combinators $\boldsymbol{P}$ such that*

$$M_1\boldsymbol{P} = N_1 \;\&\; M_2\boldsymbol{P} = N_2.$$

(ii) *There exists a combinator F such that*

$$FM_1 = \lambda xy.x \;\&\; FM_2 = \lambda xy.y.$$

(iii) *$M_1 = M_2$ is inconsistent with $\lambda\beta$.*
(iv) *$M_1 = M_2$ is inconsistent with $\lambda\beta\eta$.*
(v) *M_1 and M_2 have distinct $\beta\eta$-nfs (normal forms).*

Proof. (i)⇒(ii) Let $N_i := \lambda x_1x_2.x_i$, for $1 \le i \le 2$. By (i) there are $\boldsymbol{P}$ such that $M_i\boldsymbol{P} = N_i$. Take $F := \lambda m.m\boldsymbol{P}$.

(ii)⇒(iii) From the equation $M_1 = M_2$ one can by (ii) derive $\lambda xy.x = \lambda xy.y$, from which one can derive any equation; all derivations using just $\lambda\beta$.

(iii)⇒(iv) Trivial.

(iv)⇒(v) By the hypothesis that M_1, M_2 have β-nf and [1], Corollary 15.1.5, it follows that M_1, M_2 have $\beta\eta$-nfs. If these were equal, then $M_1 =_{\beta\eta} M_2$ and hence $M_1 = M_2$ would be consistent.

(v)⇒(i) This is the core of Böhm's theorem. A proof can be found in [1], Theorem 10.4.2. □

The equivalences do not hold for arbitrary terms M_1, M_2, not in β-nf.

Remark 1.3. Referring to Theorem 1.2 one has the following.

1. In the list of equivalences on could add (iv$^{\mathrm{a}}$) $M \neq_{\beta\eta} N$. Indeed, (iv) ⇒ (iv$^{\mathrm{a}}$) ⇒ (v).
2. The implications (i)⇒(ii), (ii)⇒(iii), (iii)⇒(iv) and (v)⇒(iv) hold trivially for all M_1, M_2. Also (v)⇒(i) holds (but not trivially), as the condition of normalizability holds by assumption.
3. In general (iv)$\not\Rightarrow$(v). One has $\Omega = \mathsf{I}$ is consistent with $\lambda\beta\eta$, as follows by the technique of [8], but Ω does not have a $\beta\eta$-nf.
4. Similarly (iv)$\not\Rightarrow$(iii). For example the set of equations

$$\{\Omega\mathsf{I} = \mathsf{U}^2_1, \Omega\mathbf{c}_1 = \mathsf{U}^2_2\}$$

is consistent with $\lambda\beta$, see [1], Corollaries 15.3.6 and 15.3.7. But the set is inconsistent with $\lambda\beta\eta$, as $\mathsf{I} =_{\beta\eta} \mathbf{c}_1$. Hence $\langle\Omega\mathsf{I}, \Omega\mathbf{c}_1\rangle = \langle\mathsf{U}^2_1, \mathsf{U}^2_2\rangle$ is consistent with $\lambda\beta$, but not with $\lambda\beta\eta$.
5. As to (iii)$\not\Rightarrow$(ii), the equation $\Omega_3 = \mathsf{I}$, with $\Omega_3 \equiv (\lambda x.xxx)(\lambda x.xxx)$, is inconsistent as shown in [8]. But if $F\Omega_3 = \lambda xy.x$ and $F\mathsf{I} = \lambda xy.y$ for some F, then by [1], Proposition 14.3.24, it follows that either Ω_3 is solvable, which it isn't, or $\forall M.FM = \lambda xy.x$, which contradicts the second equation.