Sparse Geometric Graphs with Small Dilation

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Abstract. Given a set $S$ of $n$ points in the plane, and an integer $k$ such that $0 \leq k < n$, we show that a geometric graph with vertex set $S$, at most $n - 1 + k$ edges, and dilation $O(n/(k + 1))$ can be computed in time $O(n \log n)$. We also construct $n$–point sets for which any geometric graph with $n - 1 + k$ edges has dilation $\Omega(n/(k + 1))$; a slightly weaker statement holds if the points of $S$ are required to be in convex position.

1 Preliminaries and Introduction

A geometric network is an undirected graph whose vertices are points in $\mathbb{R}^d$. Geometric networks, especially geometric networks of points in the plane, arise in many applications. Road networks, railway networks, computer networks—any collection of objects that have some connections between them can be modeled as a geometric network. A natural and widely studied type of geometric network is the Euclidean network, where the weight of an edge is simply the Euclidean distance between its two endpoints. Such networks for points in the plane form the topic of study of our paper.

When designing a network for a given set $S$ of points, several criteria have to be taken into account. In particular, in many applications it is important to

\textsuperscript{*} This work was supported by LG Electronics and NUS research grant R-252-000-166-112. Research by B.A. has been supported in part by NSF ITR Grant CCR-00-81964 and by a grant from US-Israel Binational Science Foundation. Part of the work was carried out while B.A. was visiting TU/e in February 2004 and in the summer of 2005. MdB was supported by the Netherlands’ Organisation for Scientific Research (NWO) under project no. 639.023.301.

\textsuperscript{**} National ICT Australia is funded through the Australian Government’s Backing Australia’s Ability initiative, in part through the Australian Research Council.
ensure a short connection between every pair of points in $S$. For this it would be ideal to have a direct connection between every pair of points; the network would then be a complete graph. In most applications, however, this is unacceptable due to the high costs. Thus the question arises: is it possible to construct a network that guarantees a reasonably short connection between every pair of points while not using too many edges? This leads to the concept of spanners, which we define next.

Recall that the weight of an edge $e = (u, v)$ in a Euclidean network $G = (S, E)$ on a set $S$ of $n$ points is the Euclidean distance between $u$ and $v$, which we denote by $d(u, v)$. The graph distance $d_G(u, v)$ between two vertices $u, v \in S$ is the length of the shortest path in $G$ connecting $u$ to $v$. The dilation (or: stretch factor) of $G$, denoted $\Delta(G)$, is the maximum factor by which graph distance $d_G$ differs from the Euclidean distance $d$, namely

$$
\Delta(G) := \max_{u, v \in S, u \neq v} \frac{d_G(u, v)}{d(u, v)}.
$$

The network $G$ is a $t$-spanner for $S$ if $\Delta(G) \leq t$.

Spanners find applications in robotics, network topology design, distributed systems, design of parallel machines, and many other areas and have been a subject of considerable research [3]. Recently spanners found interesting practical applications in metric space searching [14,15] and broadcasting in communication networks [19,13]. The problem of constructing spanners has received considerable attention from a theoretical perspective—see the surveys [7,17].

The complete graph has dilation 1, which is optimal, but we already noted that the complete graph is generally too costly. The main challenge is therefore to design sparse networks that have small dilation. There are several possible measures of sparseness, for example the total weight of the edges or the maximum degree of a vertex. The measure that we will focus on is the number of edges. Thus the main question we study is this: Given a set $S$ of $n$ points in the plane, what is the best dilation one can achieve with a network on $S$ that has few edges? Notice that the edges of the network are allowed to cross or overlap.

This question has already received ample attention. For example, there are several algorithms [2,12,16,18] that compute a $(1+\varepsilon)$-spanner for $S$, for any given parameter $\varepsilon > 0$. The number of edges in these spanners is $O(n/\varepsilon)$. Although the number of edges is linear in $n$, it can still be rather large both due to the dependency on $\varepsilon$ and due to the hidden constants in the $O$-notation. Das and Heffernan [4] showed how to compute, for any $c > 1$, an $O(1)$–spanner with $cn$ edges. We are interested in the case where the number of edges is close to $n$, not just linear in $n$. Any spanner must have at least $n - 1$ edges, for otherwise the graph would not be connected, and the dilation would be infinite. This leads us to define the quantity $\Delta(S, k)$:

$$
\Delta(S, k) := \min_{V(G) = S, |E(G)| = n - 1 + k} \Delta(G).
$$