Fast Edge Colorings with Fixed Number of Colors to Minimize Imbalance

Gruia Calinescu and Michael J. Pelsmajer

1 Department of Computer Science, Illinois Institute of Technology, Chicago, IL 60616, USA
calinescu@iit.edu

2 Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA
pelsmajer@iit.edu

Abstract. We study the following optimization problem: the input is a multigraph \( G = (V, E) \) and an integer parameter \( g \). A feasible solution consists of a (not necessarily proper) coloring of \( E \) with colors \( 1, 2, \ldots, g \). Denote by \( d(v, i) \) the number of edges colored \( i \) incident to \( v \). The objective is to minimize \( \sum_{v \in V} \max_i d(v, i) \), which roughly corresponds to the “imbalance” of the edge coloring. This problem was proposed by Berry and Modiano (INFOCOM 2004), with the goal of optimizing the deployment of tunable ports in optical networks. Following them we call the optimization problem MTPS - Minimum Tunable Port with Symmetric Assignments.

Among other results, they give a reduction from Edge Coloring showing that MTPS is NP-Hard and then give a 2-approximation algorithm. We give a \((3/2)\)-approximation algorithm. Key to this problem is the following question: given a multigraph \( G = (V, E) \) of maximum degree \( g \), what fraction of the vertices can be properly edge-colored in a coloring with \( g \) colors, where a vertex is properly edge-colored if the edges incident to it have different colors? Our main lemma states that there is such a coloring with half of the vertices properly edge-colored. For \( g \leq 4 \), two thirds of vertices can be made properly edge-colored.

Our algorithm is based on \( g \) Maximum Matching computations (total running time \( O(gm\sqrt{n + m/g}) \)) and a local optimization procedure, which by itself gives a 2-approximation. An interesting analysis gives an expected \( O((gn + m) \log(gn + m)) \) running time for the local optimization procedure.

Keywords: Approximation Algorithms, Graph Theory, Edge Coloring, Randomized Algorithms

1 Introduction

Berry and Modiano [2] study the benefits of using tunable electronic ports in WDM/TDM Optical Networks. They provide formulations for the “tunable”

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optimization problems of reducing the number of tunable electronic ports. These
ports are very expensive and optimal placement is very desirable.

They introduce two optimization problem. In this paper we concentrate on
the Minimum Tunable Port with Symmetric Assignments (MTPS) problem: The
input is a multigraph $G = (V, E)$ and an integer parameter $g$. A feasible solution
consists of a (not necessarily proper) coloring (called a $g$-edge coloring) of $E$
with colors $1, 2, \ldots, g$. Denote by $d(v, i)$ the number of edges colored $i$ incident
to $v$. The objective is to minimize $\sum_{v \in V} \max_i d(v, i)$.

Actually, the MTPS problem as described in [2] has a different description.
They give a non-trivial equivalence reduction to the formulation above, which
they use for proving NP-Completeness and for approximation algorithms. For
$g = 3$, they show the problem is NP-Complete by an easy reduction from Edge
Coloring in cubic graphs [7]. Indeed, one can see that a proper 3-edge coloring
(that is, a coloring with 3 colors where no two edges incident to a vertex have
the same color) of a cubic graph is the only way a 3-edge coloring can have
objective function in MTPS equal to $|V|$. The result of [7] can be used (though
we do not prove this here) to show that MTPS is APX-Hard: that is no $(1 + \epsilon)$-
approximation algorithm exists unless P=NP [1].

An edge coloring is called equitable [6] if for all vertices $v$ and colors $i, j$, we
have $d(v, i) \leq d(v, j) + 1$. It is clear that an equitable edge coloring, if it exists,
minimizes the objective function [3]. Certain classes of graphs, for example simple
graphs where no vertex has degree multiple of $g$, are known to have equitable
g-edge colorings [6].

Berry and Modiano [2] give a conceptually simple 2-approximation algorithm,
which we describe later. We give a $(3/2)$-approximation algorithm. Key to the
MTPS problem is the following question: given a graph $G = (V, E)$ of maximum
degree $g$, what fraction of the vertices can be properly edge-colored in a $g$-edge
coloring, where a vertex is properly edge-colored (or just proper) if the edges
incident to it have different colors? Our main lemma states that there is a $g$-
edge coloring with half of vertices properly edge-colored. For $g \leq 4$, two thirds
of vertices can be made properly edge-colored; this bound is tight. We leave as
an open question the problem of finding tight bounds for larger values of $g$.

Our algorithm for $g > 4$ is based on $g$ Maximum Matching computations (to-
tal running time $O(gm\sqrt{n + m/g})$) and a local optimization procedure, which
by itself gives a 2-approximation. By carefully implementing this local optimization
procedure, we prove it has an expected $O((gn + m) \log(gn + m))$ running
time. This implementation is needed to ensure the overall running time is not
dominated by local optimization. Local optimization would be a top choice of a
practitioner, and depending on the size of the instance, it may be important to
have a fast implementation. For $g = 3$ and $g = 4$ we obtain a 4/3-approximation
algorithm with running time of $O((n + m) \log n)$ and $O(n^2 + m^2)$, respectively.

A related problem was considered by Feige et. al. [5] In MAXIMUM EDGE
COLORING, given a multigraph $G = (V, E)$ and a parameter $g$, one seeks a
subgraph with maximum number of edges which can be properly edge-colored