"THE VISCOUS FLOW AROUND A CIRCULAR CYLINDER"

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1. INTRODUCTION

The problem of the viscous flow around a circular cylinder has already received large attention, therefore it might seem to be superfluous to present again a paper concerned with the same problem.

However, notwithstanding the many investigations reported so far, it cannot be said that the problem has been solved to any degree of satisfaction, neither fundamentally nor practically.

With a view to obtain more insight in these problems, at Twente University of Technology a group was formed by Mr. R.W. de Vries, Mr. H.Q.J. Meershoek, Mr. E.H. Derks and myself to attack these problems along a broad front.

Several schemes, some rather simple, were tried or tried again in order to obtain a set of overlapping results. As may be obvious some of these schemes failed while others were partially successful. In nearly all methods a trigonometric representation of the vorticity and the stream function was used, thereby reducing the partial differential equations to a set of ordinary differential equations.

At the moment the investigations continue and all that can be given now is a survey of the schemes tried and of the preliminary results obtained thereby so far.

It is only fair to say that an important stimulus to these investigations was formed by a paper of Son and Hanratty (ref. 3), since the results presented there left much to question.

2. FORMULATION OF THE PROBLEM

The partial differential equations governing the flow of a viscous fluid in two dimensions can be written in terms of the vorticity ζ and the stream function ψ, as follows

\[
\frac{\partial \zeta}{\partial t} + \frac{1}{r} \left( \frac{\partial \zeta}{\partial r} \frac{\partial }{\partial \theta} - \frac{\partial \psi}{\partial \theta} \frac{\partial }{\partial r} \right) = \frac{2}{R} \left[ \frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r^2} \frac{\partial \zeta}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \zeta}{\partial \theta^2} \right] 
\]

\[
\zeta = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} 
\]

Here R is the Reynolds number referred to the diameter of the cylinder while r and θ are cylindrical coordinates, where θ is measured from the rear of the cylinder. The boundary conditions take the following form

\[
\psi = \frac{d\psi}{dr} = 0 \quad \text{for } r = 1 
\]

\[
\zeta = 0 \quad \text{for } \theta = 0, \pi 
\]

\[
\zeta \to 0 \text{ and } \psi \to -r \sin \theta \text{ for } r \to \infty 
\]
In the case of a numerical solution based on finite difference methods, often use is made of the following transformation.

\[ r = e^{\pi \xi} \quad \eta = \theta / \pi \]  

(2.4)

This has the advantage that the right hand side of eqs (2.1) and (2.2) reduce to the Euclidean laplace operator, while the main advantage is that more points are situated in the vicinity of the cylinder, i.e. there where the change in the physical quantities is largest.

3. INVESTIGATION OF DIFFERENT SCHEMES

In this chapter the various schemes and the results thereby obtained will be discussed in some detail. It should be emphasized that the results in all cases are only preliminary and that due to further investigations some of the tentative conclusions probably will be changed.

3.1 The truncated Stokes - Picard method

This is a time independant method, hence \( \frac{\partial r}{\partial t} = 0 \). Equations (2.1) and (2.2) are reduced to very simple linear partial differential equations by adopting the following scheme

\[
R \frac{\partial}{\partial r} \left( \frac{\partial \zeta}{\partial r} - \frac{\partial \zeta}{\partial \theta} \right) - \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial \zeta}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \zeta}{\partial \theta^2} = 0
\]  

(3.1.1)

\[
\zeta^{k+1} = \frac{\partial \psi^{k+1}}{\partial r} + \frac{1}{r} \frac{\partial \psi^{k+1}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \zeta^{k+1}}{\partial \theta^2}
\]  

(3.1.2)

This system of partial differential equations is transformed into a set of ordinary differential equations by using the following representation

\[
\zeta = \sum_{n=1}^{\infty} \zeta_n \sin n\theta
\]  

(3.1.3)

\[
\psi = \sum_{n=1}^{\infty} \psi_n \sin n\theta
\]  

(3.1.4)

By this representation the symmetry condition (2.3)b is fulfilled automatically.

The left hand side of eq. (3.1.1) is represented by

\[
\sum_{n=1}^{\infty} f_n^k \sin n\theta
\]  

(3.1.5)

where \( f_n^k \) depends on all the functions \( \zeta_n \) and \( \psi_n \).

The system of the equations now becomes

\[
f_n^k = \frac{d^2 \zeta_n^{k+1}}{dr^2} + \frac{1}{r} \frac{d\zeta_n^{k+1}}{dr} - \frac{n^2}{r^2} \zeta_n^{k+1}
\]  

(3.1.6)

\[
\zeta_n^{k+1} = \frac{d^2 \psi_n^{k+1}}{dr^2} + \frac{1}{r} \frac{d\psi_n^{k+1}}{dr} - \frac{n^2}{r^2} \psi_n^{k+1}
\]  

(3.1.7)