0. INTRODUCTION.

Recall that a Banach lattice is a couple \((V, V_+)\) where \(V\) is a Banach space and \(V_+\) a cone in \(V\) defining the order of \(V\) and for which \(V\) is a lattice space. The norm and the order are related by the following axiom:

\[
\forall x, y \in V : |x| \leq |y| \Rightarrow \|x\| \leq \|y\|.
\]

This axiom implies that the lattice operations : \(x, y \rightarrow x \vee y\) or \(x \wedge y\) are continuous. Consequently, the cone \(V_+\) is closed.

The problem we are concerned with is the following : Represent \(V\) as a concrete space : \(C(T), T\) compact ; \(C_0(T), T\) locally compact ; \(L^p(T) ; 1^p, \ldots\). This is a very rich theory, where a lot has been done in the past twenty years, and I shall restrict myself to some of the most significant results.

Two kinds of representation will appear : Representation by means of continuous functions on some topological space with or without infinite values.

The kind of representation we shall obtain will depend upon the abundance of extreme generators in the cone \(P(V)\) of positive functionals on \(V\) i.e. on the abundance of real lattice homomorphisms. Without any restriction on \(V\), there won't exist any real lattice homomorphism and we shall only have Davies's representation theorem (cf. 2.4.) of \(V\) by real continuous functions on some compact space, with possible infinite values on some rare subset. On the contrary, in particular cases, such as when \(V\) is an \(M\)-space, we shall have representation theorem by real, finite-valued, continuous functions on some non-compact topological space, due to the abundance of real lattice homomorphisms. The non-compactness of the representation space is not a handicap. On the contrary, its structure expresses precise features of the Banach lattice \(V\).

References to sources and complements are relegated to the end of this paper.

1. CASE OF FINITE VALUED FUNCTIONS.

1.1. NOTATIONS.

\(V\) will be a fixed Banach lattice ; \(V_1\) denotes its unit ball ; \(V'\) is topological dual ; \(V'_1\) its dual unit ball ; \(P(V)\) the positive elements in \(V'\) and \(P_1(V)=P(V) \cap V'_1\).
An extreme generator of $P(V)$ is, by definition, a generator $D$ of $P(V)$ such that $(P(V) + D)$ is convex. If $P(V)$ has a base $B$, $D$ is extreme if, and only if, $D \cap B$ is an extreme point of $B$. $P(V)$ will denote the union of the extreme generators of $P(V)$ 
$$P_1^g = P(V) \cap P_1(V).$$

Recall that $L \in P(V)$ if, and only if, $L$ is a lattice homomorphism i.e. :
$$L(a \lor b) = \max(L(a), L(b)), (\forall a, b \in V).$$

Thus $P(V)$ is closed in $P(V)$. In particular $P_1^g$ is compact.

1.2. EXAMPLES.

If $V = C(T)$, the space of continuous real functions on a compact topological space $T$, $P(V) = \mathcal{M}_+(V)$ the cone of positive Radon measures on $T$; if $V = C_0(T)$, the space of continuous real functions on some locally compact topological space $T$ vanishing at infinity, $P(V) = \mathcal{M}_b(T)$ the cone of positive bounded Radon measures on $T$. In both cases, $P(V) \cap P(V)$ consists of the punctual measures $r \delta(t)$ where $r \in \mathbb{R}_+$ and $\delta(t)$ is the Dirac measure at the point $t \in T$. If $V = L^p(X, \theta)$, where $1 \leq p < +\infty$, and $\theta$ a positive Radon measure on some locally compact topological space, $P(V) = L^q(X, \theta) \cap P(V)$ where $q$ is the conjugate number of $p$ and $P(V) \cap P(V)$ is made of the functions with support reduced to a point of $X$ of $\theta$-measure non null.

The first theorem we state is simply a restatement of Bauer's theorem:

1.3. THEOREM.

If $V$ is an order unit Banach lattice space, there exists a compact topological space $T$ and a bipositive linear isometry of $V$ onto $C(T)$.

Let us now consider a more general case:

1.4. DEFINITION.

We say that a Banach lattice $V$ is an $M$-space if the following is true :
$$\|a \lor b\| = \max(\|a\|, \|b\|) \quad \text{for all} \quad a, b \in V_+. $$

The main interest of such spaces $V$ is given by the following result which expresses the abundance of extreme generators:

1.5. LEMMA.

If $V$ is an $M$-space, then $P_1^g$ is a cap of $P(V)$ i.e. the complement of $P_1^g$ in $P(V)$ is convex. In particular, $E(P_1^g) \subseteq P_1^g$.

From this, one gets Kakutani's theorem in a slightly modified version:

1.6. THEOREM.

Let $V$ be an $M$-space. To each $v \in V$ associate the homogeneous function $\bar{V}$ on $P(V) \cap P_1^g$ defined by :
$$\bar{V} : L \mapsto L(v) \quad \text{for all} \quad L \in P(V) \cap P_1^g.$$