ORDER IDEALS IN ORDERED BANACH SPACES

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Let \( E \) be an ordered Banach space, i.e. the positive cone \( E_+ \) of \( E \) is closed, normal and generating. A linear subspace \( I \) of \( E \) is called an order ideal if \( x \in I \), \( y \in E \) and \( 0 \leq y \leq x \) implies that \( y \in I \), i.e. if \( I_+ = I \cap E_+ \) is an extremal subset of \( E_+ \).

If \( I_+ \) generates \( I \) then \( I \) is called an ideal.

The Banach dual space \( E^* \), with the dual ordering, is also an ordered Banach space and so it is natural, and important, to study the relationship between order ideals \( I \) and their annihilators \( I^0 = \{ f \in E^*: f(x) = 0, \forall x \in I \} \).

It is easy to verify that \( I^0 \) is an order ideal whenever \( I \) is an ideal. However \( I^0 \) may be an order ideal without \( I \) being an ideal, for example when \( E = \mathbb{R}^3 \), \( E_+ = \{(x,y,z): z \geq 0, x^2 + y^2 \leq z^2\} \), and \( I \) is any two-dimensional subspace of \( E \) which intersects \( E_+ \) in an extreme ray. The precise property which \( I \) must satisfy, a kind of approximate positive-generation, is described in the following result [12].

Theorem 1. Let \( I \) be an order ideal in \( E \). Then \( I^0 \) is an order ideal in \( E^* \) if and only if \( I \) is perfect, i.e. for each \( x \in I \)

\[ \exists \text{ sequences } w_n \in I, y_n, z_n \in E \text{ such that } \|y_n\| \leq 1, \|z_n\| \leq 1 \]

\[ \text{and } -w_n + \frac{1}{n} y_n \leq x \leq w_n + \frac{1}{n} z_n, \text{ for each } n. \]

For extensive generalizations of this result see Jameson [16] and Nagel [17].

For the remainder of these notes let \( E \) be a base normed Banach space, with base \( B \) and closed unit ball \( \text{co}(B \cup -B) \). Then \( I \) is an ideal in \( E \) if and only if \( I = \text{lin} F \) for some face \( F \) of \( B \); if \( I \) is closed then so is \( F \), but the converse is much more subtle.
In fact if $F$ is closed then $\text{lin} F$ is closed if and only if each $f \in A^b(F)$ has an extension belonging to $A^b(B)$. Here we denote by $A^b(B)$ the Banach space of all bounded affine real-valued functions on $B$; this space is readily identified with $E^*$. If $K$ is a compact convex set then $A^b(K)$ is the second dual space of the ordered Banach space $A(K)$. If $F$ is a closed face of $K$ we write $F_\perp = \{f \in A(K): f(x) = 0, \forall x \in F\}$, and $(F_\perp)^\perp = \{x \in K: f(x) = 0, \forall f \in F_\perp\}$. Similarly if $F$ is a norm-closed face of $B$ we write $F_\perp = \{f \in A^b(B): f(x) = 0, \forall x \in F\}$, and $(F_\perp)^\perp = \{x \in B: f(x) = 0, \forall f \in F_\perp\}$. It is often of importance to know that $F = (F_\perp)^\perp$ or $F = (F_\perp)^\perp$; this is always the case if $F$ is finite-dimensional. However, we have the following example due to J.D. Pryce.

Example 1. Let $E = L^2[0,1]$, $F = \{f \in E: 0 \leq f \leq 1, f > 0, \|f\| \leq 1\}$, $G = \{f \in E: f \neq 0, \|f\| \leq 1\}$ and let $h \in E_+$ be essentially unbounded on $[0,1]$. Then, if $K = \text{co}(F \cup (G + h))$, $K$ is weakly compact and $F$ is a closed face of $K$, since all elements of $F$ are essentially bounded. If $\varphi \in F_\perp$ then, since $G - G$ is a neighbourhood of $0$ in $E$ and since $\text{lin} F$ is dense in $E$, it follows that $\varphi = 0$. Therefore $(F_\perp)^\perp = K \neq F$, and a fortiori $(F_\perp)^\perp = K$.

The bipolar theorem shows that if $F$ is a closed face of $K$ (or a norm-closed face of $B$) then $F = (F_\perp)^\perp = (F_\perp)^\perp$ if and only if $F = K \cap L$ ($F = B \cap L$) where $L$ is the $w^*$-closed (norm-closed) linear hull of $F$; these conditions are certainly satisfied if $L$ is $w^*$-closed (norm-closed). The following result is due to Alfsen [2] and D.A. Edwards [10].

Theorem 2. If $F$ is a closed face of $K$, then the following statements are equivalent: (i) $\text{lin} F$ is norm-closed; (ii) $\text{lin} F$ is $w^*$-closed; (iii) $\exists$ a constant $M$ such that each $f \in A(F)$ has an extension $g \in A(K)$ with $\|g\| \leq M\|f\|$.

If these statements hold then $A(K)^+|_F = A(F)^+$ if and only if $A(K)/F_\perp$ is Archimedean ordered.

Precisely analogous results hold for the space $A^b(B)$ (with the exception of (ii)).

An ideal $I$ in $A(K)$ such that $A(K)/I$ is Archimedean ordered is called an Archimedean ideal; if, in addition, $I^0$ is positively generated then $I$ is called a strongly Archimedean ideal. Since an