QUANTISATION AS DEFORMATION THEORY,

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1. Notation

We shall use the same notation and concepts as in our talk [1] of last year to this colloquium namely:

\( M \) = Riemannian configuration manifold, metric tensor \( g \), local coordinates \( q^1 \ldots q^n \).

\( T(s)M \) = space of real fully symmetric contravariant tensor fields \( S \) on \( M \), with valence \( v(S) = s \).

\( C_s(S) = s^i_1 \ldots s^i_s(q)p_{i_1} \ldots p_{i_s} \) = homogeneous function on phase space \( T^*M \) associated with \( S \).

\( [S,T]_{s+t-1}M \) = Schouten concomitant, related to Poisson bracket by \( \{C_s(S),C_t(T)\} \) = \( C_{s+t-1}([S,T]) \).

\( A = \sum_{s=0}^{\infty} T(s)M \) = Graded Lie algebra of sequences of symmetric tensors, with Schouten concomitant as Lie product.

\( H \) = Hilbert space of wave functions \( \Psi \) on \( M \).

2. Quantisation and deformation theory

Any quantisation scheme associates each \( C_s(S) \) with a Hermitian linear operator \( Q_s(S) \) on \( H \). For example we could take,

\[
\begin{align*}
\varphi & \in T(0)M, \quad (Q_0(\varphi)\Psi)(m) = \varphi(m)\Psi(m), \quad m \in M \\
X & \in T(1)M, \quad (Q_1(X)\Psi)(m) = \frac{1}{2}(-iX^i_1V_i + \text{conjugate})\Psi(m) \\
U & \in T(2)M, \quad (Q_2(U)\Psi)(m) = \frac{1}{2}(-U^i_1V_{i_1}V_{i_2} + \text{conjugate})\Psi(m) \\
S & \in T(s)M, \quad (Q_s(S)\Psi)(m) = \frac{1}{2}((-i)^s s^i_1 \ldots s^i_s V_{i_1} \ldots V_{i_s} + \text{conjugate})\Psi(m), \quad s>2
\end{align*}
\]

where \( V_i \) is the covariant derivative for the Riemannian connection on \( M \). All quantisation schemes agree with this one to leading order. With the scheme above, we find that the commutator

\[
[q_s(S),q_t(T)] = -i(q_{s+t-1}([S,T]) + q_{s+t-3}(F_1(S,T)) + q_{s+t-5}(F_2(S,T)) + \ldots)
\]
The new Lie product

\[ -C(\{s, t\})' = \{C'_a(s), C'_b(t)\}' = -C_{a+1}(\{s, t\}) - C_{a+3}(f_1(s, t)) - \ldots \]

is a deformation \[2\] of the original Poisson bracket or Schouten concomitant. All quantisation schemes furnish such deformations.

The map \( F : T^{(d)}M \times T^{(b)}M \to T^{(d+b-3)}M \) is a cocycle of order 2 in the Lie algebra cohomology of \( \mathcal{A} \). If

\[ \eta : T^{(d)}M \to T^{(d-2)}M \]

is a cochain of order 1 in \( \mathcal{A} \), and we alter some given quantisation scheme \( Q \) in (2) to another one, \( Q' \), related to \( Q \) by

\[ Q'_a(s) = Q_a(s) + Q_{d-2} (\eta(s)) \]

then the new commutation relations are

\[ \{Q'_a(s), Q'_b(t)\} = \{-i\{Q'_a(s), [s, t]\} + Q_{d+b-3}(f_1(s, t)) - \ldots \}. \]

It is well known that \( F \) is not exact, for any configuration manifold \( M \); there is no quantisation scheme in which all commutators are the quantisations of the corresponding Poisson brackets. So the problem in this formulation is which scheme to choose. Different schemes have different equations of motion. We shall scrutinize these to see if there is a "best" scheme.

3. Time development

In classical mechanics, for a system with Hamiltonian

\[ \frac{1}{2} \{ \dot{q}, q \} = C_2 (q^{-1}) \]

we have

\[ \frac{d}{dt} C_a (s) = \{C_a (s), C_2 (q^{-1})\} = C_{a+1}(s, [q^{-2}, s]). \]

In quantum mechanics, the eq. (2) is an equal time commutation relation for operators \( Q_a(s) \) whose time development is given by