Geometry of Collective Motion

D.J. Rowe, Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada

and G. Rosensteel, Department of Physics, Tulane University, New Orleans, Louisiana, 70118, U.S.A.

Two major objectives in many-body physics are: to extract a collective Hamiltonian subdynamics from the microscopic Hamiltonian equations and to realize collective states in many-particle Hilbert space. Based on the several investigations of these problems by ourselves and others [1-10], we present a systematic geometric approach to the achievement of both objectives.

The first step is to decompose N-particle configuration space, $\mathbb{R}^{3N}$, into orbits of a kinematical collective group, e.g. $SO(3)$ or $GL_{+}(3,\mathbb{R})$, and a smooth transversal. Collective motion is defined to be motion on an orbit surface and intrinsic motion, motion along a transversal.

The kinetic energy for N particles is proportional to $\Delta$, the LBO (Laplace-Beltrami operator) on $\mathbb{R}^{3N}$. Let $p_{ni} = -i\hbar \partial x^{ni}/\partial x^{ni}$, $n=1,\ldots,N,i=1,2,3$ denote the usual single-particle momenta and $\pi_{\nu} = \pi_{\nu i}(x)p_{ni}$, $\nu=1,\ldots,3N$, a set of vector fields on $\mathbb{R}^{3N}$ that form a basis of tangent vectors at each point. Then, if $g_{\mu\nu} = \sum_{n,i} \pi_{\mu i}(x)\pi_{\nu i}(x)$ is the metric and $g^{\mu\nu}$ its inverse, one can show [11] that $\Delta = \sum_{n,i} p_{ni}^{2}$ can be expressed in the form

$$\Delta = (\pi_{\mu} - i\hbar \partial_{\mu} / 3 x^{ni})g^{\mu\nu}\pi_{\nu}.$$  

Hence, to decompose $\Delta$ into collective and intrinsic components, one has simply to select a set of vector fields $(\pi_{\nu})$ which separates into a subset $(X_{1},X_{2},\ldots)$ of collective momenta, tangent to the orbits of the chosen kinematical group, and a complementary set of intrinsic momenta $(\Pi_{1},\Pi_{2},\ldots)$, orthogonal to the orbit surfaces. Then $g(X_{\mu},\Pi_{\sigma}) = 0$, all $\mu,\sigma$, i.e. $(g^{\mu\nu})$ is block diagonal, and $\Delta$ separates

$$\Delta = \Delta_{\text{coll}} + \Delta_{\text{intr}}$$

with no cross terms.

For example, if for collective quadrupole dynamics one considers the kinematical collective group $GL_{+}(3,\mathbb{R})$ and takes as collective momenta the basis $\tau_{i j} = x^{ni} p_{nj}$, $i,j=1,2,3$ of a realization of its Lie algebra, one immediately obtains

$$\Delta = \sum_{i,j,k} (\tau_{i j} - i\hbar \delta_{i j})O^{-1}k \tau_{k j} + \Delta_{\text{intr}}$$

This paper has been added to the proceedings because Dr. Rowe could not attend the conference.
where \( (Q^{-1}_{ij}) \) is the inverse of the quadrupole tensor \( Q_{ij} = \sum_{n} x^{ni} x^{nj} \).

The collective component of \( \Delta \) can be further decomposed and expressed in terms of generators of more physically interesting collective motions [6] by observing that any \( x \) on a \( GL^+(3, \mathbb{R}) \) orbit containing a fixed point \( x_0 \in \mathbb{R}^{3N} \) can be expressed

\[
x = r_1 \cdot S \cdot r_2 \cdot x_0, \quad r_1, \ r_2 \in SO(3), \ S \in \mathbb{R}^3_+,
\]

where \( \mathbb{R}^3_+ \) is the group of diagonal positive definite real 3x3 matrices and the action on \( \mathbb{R}^{3N} \) is given by the natural action on \( N \) copies of \( \mathbb{R}^3 \). Generators of motion in \( r_1, S \) and \( r_2 \) are respectively the vector fields

\[
L_A = \sum_{B,C} \varepsilon_{ABC} \sum_{i,j} R_{Bi} R_{Cj} \tau_{ij}
\]

\[
t_A = \sum_{i,j} R_{Ai} R_{Aj} \tau_{ij}
\]

\[
L_A = \sum_{B,C} \varepsilon_{ABC} \frac{\lambda_C}{\lambda_B} \sum_{i,j} R_{Bi} R_{Cj} \tau_{ij}
\]

\( (\lambda=1,2,3) \), where \( R(x) \in SO(3) \) is the rotation matrix that diagonalizes \( (Q_{ij}) \) at \( x \in \mathbb{R}^{3N} \). Specifically, if \( \lambda_1, \lambda_2, \lambda_3 \) are the principal quadrupole moments, \( t_A \) generates vibrations in \( \lambda_A \) and \( L_A \) and \( L_A \) generate \( r_1-\) and \( r_2-\)rotations, respectively, about the principal axis \( A \). The latter have sometimes been called 'vortex' rotations [6,8]. In terms of these vector fields, one readily evaluates eq.(3) to obtain

\[
\Delta_{\text{coll}} = \sum_{A} \left[ t_A \frac{1}{\lambda_A^2} - i\hbar(N-3)\frac{1}{\lambda_A^2} - 2i\hbar \sum_{B \neq A} \frac{1}{\lambda_B^2 - \lambda_A^2} t_A \right]
\]

\[+ \sum_{A} \left[ \frac{\lambda_2^2 + \lambda_2^2}{\lambda_B^2 - \lambda_C^2} (L_A^2 + L_A^2) - \frac{4\lambda_B\lambda_C}{(\lambda_B^2 - \lambda_C^2)} L_A^2 \right] (A,B,C, \text{cyclic}).
\]

As intrinsic momenta we seek a set of vector fields orthogonal to the \( GL^+(3, \mathbb{R}) \) orbit surfaces. If \( x_0 \) is some fixed non-zero vector in \( \mathbb{R}^{3N} \), it can be shown [12] that any other non-zero \( x \in \mathbb{R}^{3N} \) can be expressed

\[
x = r_1 \cdot S \cdot R \cdot x_0, \quad r_1 \in SO(3), \ S \in \mathbb{R}^3_+, \ R \in SO(N).
\]

Thus it follows that \( L_A \) generates rotations of an \( SO(3) \) subgroup of \( SO(N) \). In fact one can readily show that

\[
-L_A = J_{BC} = \sum_{m,n} p_{Bm} p_{Cn} j_{mn} \quad (A,B,C, \text{cyclic})
\]

where

\[
j_{mn} = \sum_{i=1}^{3} (x^{ni} p_{ni} - x^{ni} p_{mi})
\]