Number of eigenvalues of many-body Hamiltonians and Efimov's effect

I. M. Sigal
Department of Mathematics
Princeton University

1. In this talk I present a new method of obtaining estimates of the number of bound states of many-body systems with potentials close to critical. The estimates discussed below expose a striking phenomenon: existence of infinite number of bound states in certain short-range, many-body systems. Note that for all attractive potentials and many mixed ones the coupling constants can always be adjusted in such a way that the phenomenon occurs. The effect was experimentally observed in three nucleon scattering and in solid states. In the last case the role of particles was played by impurities. In the first case the forces between particles are close to or exactly critical, while in the second case they can easily be adjusted.

All our considerations are illustrated in the case of three particles. The Plank constant $\hbar$ is set to 1.

2. In order to formulate rigorously our results we need

Definition. A short-range, pair potential $V_\xi$ will be called critical iff

$$h_\xi \equiv (2\mu_\xi)^{-1} - V_\xi \geq 0,$$

where $\mu_\xi$ is the reduced mass for the pair $\xi$, and the equation $h_\xi \psi = 0$ has a solution of the form $\psi = (-\Delta)^{-1/2} |V_\xi|^{1/2} \phi$, where $\phi \in L^2(\mathbb{R}^3)$.

Remark. Since $h_\xi \geq 0$, such a solution is nondegenerate and is not an eigenvector of $h_\xi$ [5], we call it a quasibound state of $h_\xi$.

Theorem 1. Let $H$ be the Schrödinger operator of a three-body system with pair potentials $V_\xi \in L^p \cap L^q(\mathbb{R}^3)$, $p > 3/2 > q$. Let the masses of the particles obey the condition [7], which for the sake of space we write here only in the case $m_1 = m_j : m_i / m_k > .9$. Then if the potentials $V_{ik}$ and $V_{jk}$ approach critical points, the number of eigenvalues of $H$ increases to infinity and becomes infinite as $V_{ik}$ and $V_{jk}$ reach critical points.

Theorem 2. Let $m_k = m_j$, $m_i m_k^{-1} > .9$ and the potentials $V_{ik}$ and $V_{jk}$ be spherically symmetric and satisfy $\int_0^\infty |V_\xi(n)| n^2 dn < \infty$. Then the number of bound states (isolated eigenvalues counting multiplicities) of $H$ has the following asymptotic behavior as the potentials $V_{ik}$ and $V_{jk}$ approach critical points:

$$N = - \frac{0.0107}{\pi} (m_i / m_k - 0.9)^{3/2} \ln \rho + \text{uniformly bounded term}, \quad (*)$$
where $\rho = \max_{i,j} |\gamma_{ik}(0)|$, $\gamma_a(k) = k \cdot \cot(\delta_a(k)+\epsilon_k)$, $\delta_a(k)$ is the s-wave phase shift for the pair $a$. Note that $\gamma_a(0) < 0$ and small if and only if $V_a$ has a shallow bound state; in this case $\gamma_a(0) = -\sqrt{-\epsilon_a}$, where $-\epsilon_a$ is the bound state energy.

Corollary. If all three potentials approach critical points then Theorems 1 and 2 are true for all masses $m_1$, $m_2$ and $m_3$.

Indeed, the restriction on the masses holds always for some $(ijk) = (123)$.

Remark. An expression analogous to (*) can be obtained for a nonspherically-symmetric case as well. Here $\gamma_a(k)$ should be replaced by $\det p(1 + V_a(k^2)^{-1})$.

3. The physical idea behind our approach is that a system of three particles behaves in many ways as a system of two of the particles connected by an effective, attractive interaction, produced by the exchange of the third particle. We show that if this third particle has quasibound states with each of the other two particles, then the effective interaction is long-range, namely $= -|R|^2$, at infinity. The effective interaction described above is somewhat analogous to the interaction via an exchange by virtual particles in quantum field theory with the square root of the energy of the third particle above (times the reduced mass) playing a role of the mass of the virtual particles.

4. Below $N(A,\lambda)$ denotes the number of isolated eigenvalues (counting the multiplicities) of an operator $A$, which are less than $\lambda$ and $\Delta_x$ stands for the Laplacian in the variable $x$. A three-body Hamiltonian in the center-of-mass frame and in the Jacobi coordinates, say, for pair (12), $r = m_1 x_1 + m_2 x_2/m_1 + m_2 - x_3$, $R = x_1 - x_2$, has the form $H = -(2m)^{-1}\Delta_r - (2\mu)^{-1}\Delta_R + V$, where $m^{-1} = (m_1 + m_2)^{-1} + m_3^{-1}$, $\mu^{-1} = m_1^{-1} + m_2^{-1}$.

5. The proof of Theorem 1 is based on the following three propositions, which are given without proofs. Propositions 1 and 2 do not assume the potentials to be critical.

**Proposition 1.** Let $u: \mathbb{R}^3 \to L^2(\mathbb{R}^3)$ have an obvious smoothness, required to make the expressions below meaningful, and $\|u(R)\|_r = 1 \forall R \in \mathbb{R}^3$. Then $N(H,\lambda) \geq N(H,\lambda)$, where

$$H_u = (2\mu)^{-1}\Delta_R + \phi(R,u) \equiv \langle (H_0(R))u(R),u(R)\rangle_r$$

$$= \langle H_{BO}(R)u(R),u(R)\rangle_r - (2\mu)^{-1}\langle \Delta_R u(R),u(R)\rangle_r.$$  

Here $H_{BO}(R) = -(2m)^{-1}\Delta_r + V(R)$ is a Born-Openheimer Hamiltonian on $L^2(\mathbb{R}^3,dr)$. 