Reflection of Analytic Singularities
by
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Let $M$ be an $n$-dimensional real analytic manifold with boundary, $P$ a second order differential operator on $M$ with analytic coefficient and with real principal symbol $p$. Assume that $P$ is of principal type and that near any point of the boundary $\partial M$, there are local coordinates $(x_1,\ldots,x_n) = (x',x_n)$ such that (up to a non-vanishing factor)

$$p = \xi_n^2 + r(x',\xi')$$

where $\partial x', \xi', r(x',0,\xi')$, $\xi_j dx_j$ are linearly independent. This condition can be formulated invariantly ([5]), and is satisfied by the wave operator in a cylinder, $\mathbb{R}_t \times \Omega_x$, $\Omega \subset \mathbb{R}^{n-1}$.

We decompose

$T^*\partial M \setminus \emptyset = \mathcal{E} \cup G \cup \mathcal{K}$

where $\mathcal{E}, G, \mathcal{K}$ are given by $r_0 > 0$, $r_0 = 0$, $r_0 < 0$ respectively, and $r_0(x',\xi') = r(x',0,\xi')$. At a point of $G$, the complement $C_M$ is strictly convex along the bicharacteristics iff $\partial r/\partial x_n < 0$ and we let this inequality define a subset $G_+ \subset G$. Let

$\Sigma_b = \mathcal{K} \cup G \cup \left( p^{-1}(0) \big|_M \right)$

with the natural topology.

Definition. An analytic ray is a continuous curve $\gamma:I \rightarrow \Sigma_b$, where $I \subset \mathbb{R}$ is an interval, such that for every $t_0 \in I$:

1° If $\gamma(t_0) \in p^{-1}(0) \big|_M$, then $\gamma$ is differentiable at $t_0$ and

$$\dot{\gamma}(t_0) = H_p(\gamma(t_0)),$$

where $H_p$ is the Hamilton field of $p = \xi_n^2 + r(x',\xi')$.

2° If $\gamma(t_0) \in \mathcal{K}$ then $\gamma(t) \in p^{-1}(0) \big|_M$ for $|t-t_0| > 0$ small enough.

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If $\gamma(t_0) \in G$ and we write $\gamma(t) = (x(t), \xi(t))$, then 

$(x(t), \xi'(t))$ is differentiable at $t_0$ and $\dot{x}_2(t_0) = 0$, 
$(\dot{x}'(t_0), \xi'(t_0)) = H_{r_0}$.

Recall from Melrose-Sjöstrand [6], that the corresponding definition for $C^\infty$-rays is obtained by replacing $H$ by $H \cup G_+$ in $2^\circ$. Thus contrary to a $C^\infty$-ray, an analytic ray may glide along a strictly convex part of an obstacle (thinking of the wave operator in a cylinder as our main example).

We write 

$u \in \mathcal{D}'(M)$, if $u$ is an extendable distribution on $\bar{M}$, 
$u \in a(M)$, if $u$ is real-analytic near $M$, 
$v \in a(\partial M)$, if $v$ is real-analytic on $\partial M$.

If $u \in \mathcal{D}'(M)$, $Pu \in a(M)$ we define

$$WF_{ba}(u) = WF_a\left(u|_M\right) \cup \left(WF_a\left(u|_{\partial M}\right) \cup WF_a\left(x_n u|_{\partial M}\right)\right)$$

$$\subset \{T^*\bar{M}\setminus 0\} \cup \{T^*\partial M\setminus 0\},$$

where $WF_a$ denotes the analytic wavefront set.

If $u \in \mathcal{D}'(M)$ and

$$(1) \quad Pu \in a(M), \quad u|_{\partial M} \in a(\partial M)$$

it follows from results of Schapira [10] that $WF_{ba}(u)$ is a closed conic subset of $\Sigma_b$.

Our main result is,

Theorem 1. ([11].) If $u \in \mathcal{D}'(M)$ satisfies (1) and $\rho \in WF_{ba}(u)$, then there exists a maximally extended analytic ray passing through $\rho$ and contained in $WF_{ba}(u)$.

In $T^*\bar{M}\setminus 0$ this is due to Sato-Kawai-Kashiwara [9], Hörmander [3] and near $K$ the result is due to Schapira [10].