Abstract. The border rank $t_B$ over the field $F$ of the non degenerate $p \times q \times 2$ tensor $A = [B, C]$ is such that $\max (p, q) \leq t_B \leq \max (p, q) + \delta$, with $\delta = 0$ if the invariant polynomials of $B + \lambda C$ have roots in the closure of $F$, $\delta = 1$ otherwise. A pair of non degenerate $p \times q$ bilinear forms can be approximated with at most $\max (p, q) + 1$ non scalar multiplications over any field.

1. Introduction.

The problem of computing a set of bilinear forms with the least number of non scalar multiplications (1) over a field $F$ is connected to a problem of linear algebra, namely to find the rank of a third order tensor $A$. In few cases this problem has been solved completely, see for example [9] concerning Toeplitz matrices, [12], [14] for $2 \times 2$ matrix multiplication, [7] for the computation of quaternions product.

Recently Ja'Ja' [8] has faced, analyzed and completely solved a problem which includes a wider class of cases, namely the problem of the evaluation of a pair of bilinear forms. In other words the rank of a $p \times q \times 2$ tensor has been completely determined in terms of the elementary divisor of a suitable matrix.

In a recent paper [4] it has been pointed out that a set of bilinear forms can be evaluated with a multiplicative complexity lower than the rank of the associated tensor by allowing an arbitrarily small error. The complexity of such an approximate algorithm equals the border rank of the associated tensor (see sect. 2). Approximate algorithms and the border rank have been used to

(1) A multiplication $a \times b$ is non scalar over $F$ if $a, b \notin F$. For the sake of brevity we use the term "multiplication" instead of "non scalar multiplication".
approximate nxn matrix product with $O(n^{\alpha})$ complexity, $\alpha \approx 2.7799$ [3], to compute nxn matrix product with $O(n^{\alpha} \log n)$ complexity, $\alpha \approx 2.7799$ [2], $\alpha \approx 2.52$ [11], [10].

In this paper we consider the problem of finding the border rank of a pxq2 tensor over a field $F$ using the Kronecker's theory of pencils. Our main result is expressed by proposition 4.4, namely the border rank $t_B$ of $A=[B,C]$ is such that $\max(p,q) \leq t_B \leq \max(p,q) + \delta$ with $\delta = 0$ if the polynomial $\det(B+\lambda C)$ has roots in the closure of the field $F$, $\delta = 1$ otherwise, where $B+\lambda C$ is the regular kernel of the pencil $B+\lambda C$. An analogous result holds when $F$ is finite. From the computational complexity point of view this fact implies that any pair of pxq bilinear forms can be approximated over $F$ by $\max(p,q)+1$ multiplications. Other consequences are exposed in section 3, (namely: the product of two polynomial modulo a n-degree polynomial can be approximated with n multiplications over $C$; the product of complex numbers cannot be approximated with less than three multiplications over the real field $R$; the result about a topological property of triangular Toeplitz matrices, exposed in [1] is improved in a very simple way).

2. Definitions and notations.

Let $F$ be a numeric field, of cardinality $f$. If $f=\infty$ we endow $F$ with the topology induced by the distance $d(x,y) = |x-y|$, $x,y \in F$ and we set $\bar{F}$ for the topological closure of $F$ so that we have $R = \bar{F}$ where $Q$ and $R$ are respectively the rational and the real field. We denote with $C$ the complex field.

Let $F[\xi]$, $F(\xi)$ be respectively the ring of polynomials over $F$ and the field of rational functions over $F$ in the variable $\xi$.

Let $A = \{a_{ijk}\}$, $a_{ijk} \in F$, $i=1,\ldots,p$, $j=1,\ldots,q$, $k=1,\ldots,m$, be a pxqmn third order tensor. $A$ is a degenerate tensor if one of the following three sets of matrices $\{A_k\}$, $\{B_j\}$, $\{C_i\}$ consists of linearly dependent matrices. The corresponding sets of bilinear forms $\{x^TA_k y: k=1,\ldots,m\}$, $\{x^TB_j z: j=1,\ldots,q\}$, $\{y^TC_i z: i=1,\ldots,p\}$ are said to be degenerate if $A$ is degenerate. The matrices $A_k$, $B_j$, $C_i$ are the slabs of $A$ and when it occurs we denote $A$ by its slabs, namely $A = [A_1,\ldots,A_p]$ let us recall some definitions and properties of the rank and the border rank of a tensor [4].