INTRODUCTION

Quantum field theorists have recently been interested in calculations of the following type. Let $M$ be a multiply connected Riemannian manifold. Assume the Hamiltonian $\hat{H}$ is intrinsically defined on $M$. Let

$$Z(\beta) = \text{Tr}(e^{-\beta \hat{H}})$$

be the partition function on $M$. Then the effective Lagrangian density $L^{(1)}$ on $M$ is determined by

$$L^{(1)} = \frac{1}{\beta} \int_0^\infty \frac{d\beta}{\beta} Z(\beta) e^{-\beta \overline{\mu}^2 / \text{Vol}(M)}.$$

Of course, the partition function is of interest in its own right from the point of view of quantum-statistical mechanics on $M$. This paper will concern itself with a review of three important concepts: partition functions, asymptotics, and scattering theory. The calculation of the partition functions in simple cases for compact spaces in quantum field theory and quantum statistical mechanics is demonstrated in Secs. 4 and 5. The general result is presented in Sec. 6 for compact quotients of noncompact symmetric spaces of rank one. The traces are evaluated in terms of the density matrices.

When the spaces are no longer compact, the density matrix is no longer of trace class. As we motivate in Sec. 2 from scattering theory, the "noncompact" part of the density matrix must be subtracted off to obtain a well-defined partition function. This is precisely what happens in the finite-volume theory presented in Sec. 7. The noncompact term in that case is expressed in terms of the theory of Eisenstein series and this is where scattering theory plays the crucial role. The class of spaces that we are ultimately led to consider are Riemannian locally symmetric spaces with nonpositive curvature. Thus $M = \Gamma \backslash G/K$ where $G$ is a real connected noncompact semisimple Lie group with finite center (e.g. $SO_0(n,1)$), $K$ is a maximal compact subgroup and $\Gamma$ is a discrete subgroup of $G$ which we shall assume is without torsion and $\Gamma \backslash G$ compact in Sec. 6 and such that Vol $(\Gamma \backslash G)$ is finite in Sec. 7.

1. PARTITION AND DENSITY FUNCTIONS

The partition function is determined by the density matrix $\rho$ which is the solution of the Bloch equation on $M$:

$$\frac{\partial \rho}{\partial \beta} + \hat{H}\rho = 0,$$

where $\rho(x'',x',0) = \delta(x''-x')$. The density matrix on $M$ and its simply connected covering space $\tilde{M}$ are related by

$$\rho(x'',x',\beta) = \sum_{\gamma \in \Gamma} a(\gamma) \tilde{\rho}(x'',x',\beta),$$

where $a$ is a unitary one-dimensional representation of $\Gamma$.

If $\hat{H}$ has a discrete one set of eigenvalues $\lambda_n$,

$$\hat{H} |\tilde{n}\rangle = \bar{\lambda}_n |\tilde{n}\rangle,$$
then formally we have

$$\tilde{\rho} = e^{-\beta \hat{H}} = \sum_n e^{-\beta \hat{\xi}_n} |\tilde{n} > < \tilde{n}|$$

and

$$Z(\beta) = \text{Tr}(e^{-\beta H}) = \sum_n e^{-\beta \xi_n}.$$ 

2. SCATTERING PHASE SHIFT

For scattering theory problems involving a central potential $V(r)$, the renormalized partition function

$$Z_L(\beta) = \text{Tr}(e^{-\beta H_L} - e^{-\beta H_f})$$
is related to the phase shift $\eta_L(k)$ by

$$Z_L(\beta) = \frac{1}{\pi} \int_0^\infty e^{-\beta k^2} \frac{d\eta_L(k)}{dk} dk,$$

where $H_L = H_f + V(r)$ and $H_f = \frac{\hbar^2}{2m} \left( \frac{d^2}{dr^2} - L(L + 1)r^{-2} \right)$.

The scattering length $A_L$ is given by the "low temperature" asymptotics on $Z_L(\beta)$. Here, $\eta_L(k) \sim -A_L k^{2L+1} (\text{mod } \pi)$ as $k \to 0$, and $A_L = (-2m/\hbar^2) \lim_{\beta \to \infty} \frac{1}{\beta} \frac{d}{d\beta} \ln Z_L(\beta)/a(L)$, where $a(L)$ is a known constant.

3. ZETA FUNCTION RENORMALIZATION

Zeta-function renormalization in quantum field theory treats the problem of finding the effective energy momentum tensor $< T_{\mu\nu} >$ or the total energy $E = -\int_M L^{(1)}$ (or the integrated form of $< T_{\mu}^{\mu} >$), in terms of the zeta functions $\zeta(s)$. The connection between $\zeta(s)$ and the density matrices $\rho(\beta)$ is given by the Mellin transform

$$\zeta(s, x', x) = \frac{1}{\Gamma(s)} \int_0^\infty \rho^{-1} \beta^{s-1} \rho(\beta, x', x') \beta^{-s} = \sum_{\lambda} \eta_{\lambda^a} \lambda^{-1}$$

in the case of a discrete spectrum, where $\eta_{\lambda^a}$ is the multiplicity of the representation $a$ in the $\lambda^a$-representation. One of the standard results of zeta function renormalization is that the one-loop effective action $W^{(1)}$ is given by

$$W^{(1)} = -\lim_{\beta \to \infty} \frac{1}{2} \left[ \frac{\zeta(0)}{s-1} + \zeta'(0) \right].$$

4. PLANAR RIGID ROTATOR

The partition function of the planar rigid rotator ($H = -\frac{\hbar^2}{2m} \left( \frac{2\pi}{L} \right)^2 \frac{d^2}{d\theta^2}$, $M = S^1$) is

$$Z(\beta) = 1 + 2 \sum_{n=1}^{\infty} \exp \left[ -\beta \frac{\pi^2}{2m} \left( \frac{2\pi}{L} \right)^2 m^2 \right],$$

which is twice the classical theta function. The Mellin transform of $[Z(\beta) - 1/2]/2$ is the Riemann zeta function

$$\zeta(2s) = \sum_{n=1}^{\infty} n^{-2s}.$$