1. INTRODUCTION

The aim of this paper is to discuss methods for the full approxima-
tion of combinatorial problems and to study the full approximability
of a class of NP-complete optimization problems defined over a lattice.
Most combinatorial optimization problems can be naturally defined as
optimization problems over lattices according to the ground algebraic
structure of the set of feasible solutions. For example the problem
of graph colouring can be viewed as an optimization problem over a
partition lattice, the problem of minimum spanning tree is an optimi-
zation problem over a matroid, etc.. In [AMP] a large class of optimi-
zation problems was formalized as the class of max-subset problems
over power sets and some basic properties of these problems were stu-
died. Despite its simple characterization the class of max-subset
problems is indeed sufficiently general to include problems with very
different properties with respect to approximability. In fact this
class includes many problems which are known to be non fully approxi-
mable and, at the same time, practically all known examples of fully
approximable NP-complete problems.

The existence of good approximations to the solution of hard
optimization problems has been studied by several authors [S,GJ,JK,L,
etc.]. What is more interesting for the development of our work is
that 1) the techniques used in proving the full approximability of a
problem are essentially based on variations of dynamic programming,
2) generally single problems (and not classes of problems) have been
shown to be fully approximable.

In particular as regards 2) many difficulties arise when trying
to find general natural conditions for the approximability and despite
of the interest for this type of results few steps have been made in
this direction ([PM],[KS]).

In order to establish a connection between good approximability
of hard problems and the intrinsic combinatorial properties which
characterize such problems it is useful to restrict ourselves to con-
sidering max-subset problems and the properties of the set of their feasible solutions.

In the whole we can say that three possible research areas are worth-while of being pursued: 1) to find new simple methods of full approximation, 2) to give general conditions for the full approximability of a class of problems, 3) to introduce new approximate algorithms of lower complexity for problems which are already known to be fully approximable.

In this paper we will be concerned with points 1) and 2). In fact, in par. 3 we will consider a new method for showing the full approximability. Its computational complexity will be studied and its advantages with respect to the classical schemes will be also shown. Instead in par. 4 we will give a sufficient condition for the full approximability of a subclass of max-subset problems which is based on the structural properties of the set of feasible solutions and which is verified by the most important problems which are known to be fully approximable.

2. A FULLY POLYNOMIAL APPROXIMATION SCHEME

Given an NP-complete optimization problem $A$ with measure $m$ the following definitions capture the concept of good approximability.

**DEFINITION 2.1.** A is an $\varepsilon$-approximate algorithm for $A$ if, given any instance $x \in A$, we have

$$\frac{|m^*(x) - m(A(x))|}{m^*(x)} \leq \varepsilon$$

where $m^*(x)$ is the measure of the optimal solution of the instance $x$.

**DEFINITION 2.2.** A problem $A$ is said to be

a) polynomially approximable if given any $\varepsilon > 0$ there exists an $\varepsilon$-approximate algorithm for $A$ which runs in polynomial time;

b) fully polynomially approximable if $A$ is polynomially approximable and there exists a polynomial $q$ such that, given any $\varepsilon$, the running time of the $\varepsilon$-approximate algorithm is bounded by $q(|x|, 1/\varepsilon)$.

**DEFINITION 2.3.** A constructive method that, for any given $\varepsilon$, provides the corresponding polynomial $\varepsilon$-approximate algorithm $A^\varepsilon$ is said to be a polynomial approximation scheme (PAS). Besides if, for every $\varepsilon$, the running time of $A^\varepsilon$ is bounded by $q(|x|, 1/\varepsilon)$ for some polynomial $q$ we say that the scheme is a fully polynomial approximation scheme.

As we said in the introduction the main aim of this paper is to