OBSERVABILITY AND NERODE EQUIVALENCE IN CONCRETE CATEGORIES

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Abstract. Functorial automata are studied in a concrete category $K$ with structured hom-sets. For each functor $F : K \rightarrow K$, which respects this structure, the observability morphisms of $F$-automata are defined analogously to those of sequential automata. If each $F$-automaton has an observable reduction, the minimization problem is both much simplified (in fact, translated to the image factorization of the observability morphisms) and made global. We prove that this is the case iff each behavior has a Nerode equivalence. First, we present a survey of the related results on minimal realization and Nerode equivalence.

I. Minimal realization (a survey)

I.1 Let $K$ be a category and $F : K \rightarrow K$ a functor. An $F$-algebra is a pair $(Q, \delta)$ consisting of an object $Q$ in $K$ and a morphism $\delta : FQ \rightarrow Q$. An $F$-automaton is an $F$-algebra with an object $Y$ and an (output) morphism $y : Q \rightarrow Y$. An initial $F$-automaton has, moreover, an object $I$ and an (initialization) morphism $i : I \rightarrow Q$. This is the concept introduced by M. A. Arbib and E. G. Manes [6]; all the notions of this section are from [6].

Examples. (i) Sequential $\Sigma$-automata are $F$-automata in the category of sets, $K = \text{Set}$, where $F = S_\Sigma$ is the following functor

$$S_\Sigma Q = Q \times \Sigma$$

and

$$S_\Sigma f = f \times \text{id}_\Sigma$$

(for each set $Q$ and each map $f$). Here $I$ is a singleton set mapped by $i$ to the initial state.

(ii) Linear sequential $\Sigma$-automata, where $\Sigma$ is a module over a commutative ring $R$, are $F$-automata in the category of $R$-modules, $K = R \text{- Mod}$. Here, again, $F = S_\Sigma$ with $S_\Sigma Q = Q \times \Sigma$ and
The morphism \( \delta : Q \times \Sigma \to Q \) here decomposes as follows:
\[
\delta(q, \sigma) = \delta_1(q) + \delta_2(\sigma) \quad \text{(for all } q \in Q; \sigma \in \Sigma) \]
for unique linear maps \( \delta_1 : Q \to Q \) and \( \delta_2 : \Sigma \to Q \).

(iii) Bilinear sequential \( \Sigma \)-automata are \( F \)-automata in \( R \)-Mod where \( F = V_{\Sigma} \) is the tensor-product functor:
\[
V_{\Sigma}Q = Q \otimes \Sigma \quad \text{and} \quad V_{\Sigma}f = f \otimes \text{id}_\Sigma.
\]

1.2 A homomorphism from an \( F \)-algebra \((Q, \sigma)\) into an \( F \)-algebra \((Q', \sigma')\) is a morphism \( f : Q \to Q' \) in \( K \) with \( f \cdot \sigma = \sigma' \cdot Ff \).

A morphism of automata is a homomorphism commuting with the outputs \((y = y' \cdot f)\) and, for initial automata, with the initializations \((i = i' \cdot f)\).

A \textit{reduction} of an \( F \)-automaton \( M \) is an \( F \)-automaton \( M' \) together with a morphism \( f : M \to M' \) which is a regular epi (a coequalizer) in \( K \). A reduction \( f_0 : M \to M_0 \) is \textit{minimal} if each reduction \( f : M \to M' \) of \( M \) can be further reduced to \( M_0 \), i.e., there exists \( h : M' \to M_0 \) with \( f_0 = h \cdot f \).

We shall assume that \( K \) has regular factorizations, i.e., each morphism \( f \) factorizes as \( f = m \cdot e \) where \( e \) is a regular epi and \( m \) is a mono.

1.3 A \textit{free} \( F \)-algebra generated by an object \( I \) in \( K \) is an \( F \)-algebra \((I^\#, \eta)\) together with a universal morphism \( \eta : I \to I^\# \).

The universality means that for each \( F \)-algebra \((Q, \sigma)\) and each morphism \( f : I \to Q \) there exists a unique homomorphism
\[
f^\# : (I^\#, \eta) \to (Q, \sigma)
\]
with \( f = f^\# \cdot \eta \). The functor \( F \) is called a \textit{varietaor} if \( I^\# \) exists for each \( I \).

Examples. (i) The functor \( S_{\Sigma} : \text{Set} \to \text{Set} \) is a varietaor. The free algebra on one generator (say, \( 0 \)) is the string algebra \( \Sigma^* \) with
\[
f : \Sigma^* \times \Sigma \to \Sigma \quad ; \quad (\sigma_1, \ldots, \sigma_n; \sigma) \mapsto (\sigma_1 \ldots \sigma_n \sigma)
\]