QUASI-EQUATIONAL LOGIC FOR PARTIAL ALGEBRAS

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Partial algebras occur quite often in connection with the foundations of Computer Science. Even many-sorted algebras are quite often partial ones at the first glance and only afterwards they are usually completed (as many-sorted algebras) by so called "error values". In this note we want to show, that partial algebras can quite easily be treated by model theoretic methods as structures independently from any possible completion. It is one in a sequence of papers with the subtitle: "A unifying approach towards a two-valued model theory for partial algebras" (cf. [ABN 80] and [B 81]).

We start here by considering special axiomatizable classes of partial algebras in which free (partial) algebras still exist with respect to any set, and universal solutions of arbitrary partial algebras exist whenever this structure allows any homomorphism into at least one object of the class at all. All the results which we will state can be proved without using the Axiom of Choice (cf. [ABN 80]).

We will restrict our considerations to the one-sorted case in order to keep the representation simpler; but everything can easily be transferred to many-sorted partial algebras as long as all sorts of the structures under consideration are supposed to be nonvoid (cf. [GoM 81]) for many-sorted algebras with possibly void sorts).

One concern of this note is also, to provide a common background for all the so called notions of validity of equations in partial algebras which are floating around, and to show that all but one are implications of very special structure in each case.

1. Motivation:

When dealing with algebraic structures, the notions of terms and equality between terms play a very important rôle, and for a model theoretic approach they are fundamental. But so far at least algebraists could not decide for the most appropriate concept of equality in connection with partial algebras. The reason is that as soon as one wants to say something about equality of terms in connection with partial algebras, one has also to say something about the existence of the interpretation of the terms involved, and that is the point where the "algebraic" approaches differ, as we shall make more precise
later (cf. [SI 68], [Hö 73] or [J 78], also [Ed 73]). But also the
approaches proposed so far by logicians (at least those of which we
know) are not satisfactory as far as they immediately use a three-
valued logic, whence they cannot express the non-existence of the
interpretation of a term in the object language (cf. [E 69] and the
references there). But it is often quite useful to be able to formu-
late axioms which express just the non-existence of the interpretation
of a term. Since this is possible in the approach which we are going
to propose (and for some other logical reasons), and since this is
done through the semantics of our notion of equality, we want to speak
about existence-equality (briefly: E-equality) and existence-equations
(briefly: E-equations) in order to stress this new quality of our
semantics. For the same reason we shall use the symbol " ≠ " instead
of " = ", in order to indicate E-equality.

2. The language:
Actually we use the same language with terms and equality which one
usually considers in connection with total algebras of some given type.
The only difference is the notation "E-equality" and the symbol " ≠ ":

2.1. Definition of terms. Let Ω be any set, the set of operation sym-
boles, and Δ := (nf) f ∈ Ω the corresponding type, i.e. a family of natu-
ral numbers, such that nf is the arity of the operation symbol f.
Moreover let V be any denumerable set of variables. Then terms of type
Δ with variables in V are defined as follows:
(i) Each variable v ∈ V is a term.
(ii) If f ∈ Ω is any operation symbol, and if t0, ..., tn−1 are
terms, then ft0...tnf−1 is a term.
(iii) Only such sequences are terms, which are formed according to
(i) and (ii). We denote the set of all terms by T.

2.2. Definition of formulas. Let Ω, Δ, V and T be given as in Defini-
tion 2.1. A formula corresponding to the set T of terms is then de-
defined as follows:
(i) If t and t' are any terms, then the E-equation t ≡ t' is a for-
mula.
(ii) If φ and ψ are any formulas, then ¬φ, (φ ∧ ψ), (φ ∨ ψ), (φ → ψ),
and (φ ↔ ψ) are formulas.
(iii) If φ is any formula, and if v is any variable, then (∀v)φ,
and (∃v)φ are formulas.