Abstract: By giving a matrix inversion algorithm that uses a small amount of space, a result of Simon is improved: For constructible functions $f(n) \notin o(\log n)$ $f(n)$ tape-bounded probabilistic Turing machines can be simulated on deterministic ones within $(f(n))^2$ space.

1. Introduction

The comparison of the relative power of deterministic, nondeterministic and probabilistic algorithms has become a central problem of theoretical Computer Science. These questions seem to be very important and very difficult to answer. The aim of this work is to improve a result of Simon [5] on the relations between tape-bounded probabilistic, and deterministic Turing machines. Using a fast matrix inversion routine on MRAMs, Simon proved that $f(n)$ tape-bounded probabilistic Turing machines can be simulated within $(f(n))^6$ deterministic tape.

Contrary to Simon we want to give a directed simulation of tape-bounded probabilistic Turing machines on deterministic ones with squared amount of tape. On the one hand our result states a strong connection to Savitch [4], on the other hand the theorem does not follow from Savitch's result. In the proof we follow the ideas of Simon up to the moment when he passes over to the computation on MRAMs.

Assuming a standard Turing machine model [3], we recall some definitions. A probabilistic Turing machine (PTM) is a pair consisting of a nondeterministic Turing machine called the underlying machine, where at most two configurations may succeed any configuration and an unbiased coin. In every time the next configuration is determ-
ined by the underlying nondeterministic Turing machine and the result of tossing the coin.

Given a PTM $M$ and an input $w$ we say that $M$ accepts $w$ if the probability $P_M(w)$ that $M$, with input $w$, enters an accepting state is greater than $\frac{1}{2}$.

The language accepted by $M$ is $L(M) = \{w : P_M(w) > \frac{1}{2}\}$. A probabilistic Turing machine is $f(n)$ tape-bounded if the underlying nondeterministic machine is $f(n)$ tape-bounded. We abbreviate the classes of languages acceptable on $f(n)$ tape-bounded PTM with $\text{Pr SPACE}(f(n))$ and the classes of languages acceptable on $f(n)$ tape-bounded (non-)deterministic Turing machines with $\text{DSPACE}(f(n))$ (resp. $\text{NSPACE}(f(n))$).

The main theorem is now formulated as follows:

**Theorem:** $\text{Pr SPACE}(f(n)) \subseteq \text{DSPACE}(f^2(n))$, for all constructible $f(n) \in O(\log m)$

As $\text{Pr SPACE}(f(n)) \subseteq \bigcup_c \text{DTIME}(2^c f(n))$ follows immediately from Simon's proof [5] and because $\bigcup_c \text{DTIME}(2^c f(n)) = \text{ASPACE}(f(n))$ [2] ("A" stands for Alternating Turing machine) we get as a final corollary:

$\text{DSPACE}(f(n)) \subseteq \text{NSPACE}(f(n)) \subseteq \text{Pr SPACE}(f(n)) \subseteq \text{DSPACE}(f^2(n))$ and $\text{Pr SPACE}(f(n)) \subseteq \text{ASPACE}(f(n))$, for all constructible $f(n) \in o(\log n)$.

Further, our result allows an improvement of [6], i.e. the deterministic simulation of an $f(n)$ tape-bounded probabilistic Turing machine transducer can be done within $f^4(n)$ space, whereas in [6] $f^{36}(n)$ space is needed.

2. Algorithms for the multiplication of matrices and integers which need a small amount of space

Let $p$ be any prime

**Lemma 1:** The product of $m \times n \times x$ matrices over a finite field with characteristic $p$ can be computed on a deterministic Turing machine within $O(\log m \cdot \log (m + n + p))$

**Proof:** Given $n \times n$ matrices $A_1, A_2, \ldots, A_m$. For $1 \leq i, j \leq n$ we want to