In this talk I present results from [4] on the phase diagrams of two-dimensional $P(\phi)_2$ models at low temperatures. Let $P = P(\mu_1, \ldots, \mu_{r-1})$ lie in an $(r-1)$-dimensional space of perturbations of a polynomial with $r$ degenerate minima. The low temperature limit is the scaling $P(\phi) \to \lambda^{-2}P(\lambda\phi)$ with $\lambda \ll 1$. For small $\lambda$, I can construct $k$ distinct states on $\binom{k}{k}$ hypersurfaces of codimension $k-1$ in the space of perturbations (the space of the $\mu$'s). For $r = 3$, for example, $P$ and its phase diagram might look like

Each state is a Wightman field theory with mass gap, and $\langle \phi \rangle \sim \lambda^q$ in the $q$th state ($q = 1, 2, 3$).

The scaling $P(\phi) \to \lambda^{-2}P(\lambda\phi)$ leaves quadratic terms invariant, and suppresses cubic and higher order terms with powers of $\lambda$. Separations between minima grow as $\lambda^{-1}$ and heights of potential barriers grow as $\lambda^{-2}$. Thus perturbative corrections to the Gaussian are small, and nonperturbative effects arising from fluctuations between minima are small. These are the essential criteria for the convergence of a cluster expansion. In fact the cluster expansion is used not just to prove a mass gap, but also in an essential way to construct the phase diagram.

The use of expansion techniques to determine a phase diagram is new to quantum field theory. Previous work has used symmetries or correlation inequalities and other techniques valid for special polynomials. The pioneering work in this area was that of Glimm, Jaffe, and Spencer [3], who invented the mean field cluster expansion for the $\lambda\phi^4 - \frac{1}{4}\phi^2 - \mu\phi$ model. Gawędzki [2] found a point where three distinct phases coexist for a certain $\phi^6$ model. Summers [6] mapped out the remainder of the phase diagram of that model. Balaban and Gawędzki [1] constructed two states for the low temperature pseudoscalar Yukawa model. In contrast to these works, the method I will describe applies to essentially arbitrary (low temperature) polynomials. The technique owes a great deal to the work of Pirogov-Sinai [5], who constructed the phase diagrams of low temperature lattice statistical mechanics models where the spins take finitely many values.
Let me now state the main result.

**THEOREM.** Let \( P_1(u_1, \ldots, u_r) \) be a polynomial that depends smoothly on \( u \) in a neighborhood of the origin in \( \mathbb{R}^{r+1} \). Suppose \( P_1 \) has minima at \( u_1 < \ldots < u_r \) and \( P''(u_q) = \frac{m_q^2}{2} \neq 0 \). With \( E_q = P_1'(u_q) \), suppose that \( E_q = E_r' \), at \( u = 0 \) and that

\[
\det \left( \frac{\partial (E_q - E_r')}{\partial u_j} \right)_{i,j=1,\ldots,r-1} \neq 0.
\]

(These conditions avoid near-critical polynomials and ensure that the parameters \( u_i \) break the degeneracy of \( P_1 \) properly.) Certain lower bounds on \( P_1 \) are assumed to prevent any minimum from dominating too strongly over the others. Put \( P(\phi) = \lambda^{-2} P_1(\lambda \phi) \) and assume \( \lambda \ll 1 \).

Define finite volume expectations from the interaction

\[
\mathbb{V}_q = \int \mathbb{P}(\phi) - \lambda \int \left( \phi - \xi_q \right)^2 - E_q(\phi); \\
\mathbb{V}_q = \mathbb{E}_q = \frac{1}{2^q} \int e^{-\lambda \int \phi - \xi_q} d\mu_q(\phi - \xi_q)
\]

Here \( d\mu_q(\phi - \xi_q) \) is the Gaussian measure in which \( \phi \) has mean \( \xi_q \) and covariance \( (-\lambda^2 + m_q^2) \mathbb{I}_q \). Then for each \( \mu \) there is a set of \( \theta \)'s such that the \( \lambda \to \mathbb{R}^2 \) limit exists and yields a Wightman quantum field theory with \( \langle \phi \rangle \approx \xi_q \) and with mass gap. The stable \( \theta \)'s, as these will be called, are given by a phase diagram homeomorphic to the \( \lambda = 0 \) phase diagram. Thus there exists at least \( k \) distinct states on \( \mathbb{R}^{r-1} \) hypersurfaces of codimension \( k-1 \) in \( \mathbb{R}^{r-1} \).

Remark. It is an open question, even on the lattice, whether these are the only states and whether expectations with unstable boundary conditions converge to the stable states as \( \lambda \to \mathbb{R}^2 \).

The crux of the proof is to show that for a stable phase \( \phi \), the ratio of partition function estimates

\[
\frac{Z(\mathbb{V}^m)}{Z(\mathbb{V}^2)} \leq e^{c|\mathcal{V}|}
\]

holds for all \( m \in \{1,\ldots,r\} \) with \( c \) independent of \( \lambda \) and \( \mathcal{V} \subseteq \mathbb{R}^2 \). Here \( Z(\mathbb{V}^m) \) is the partition function with boundary condition \( \phi = \xi_m \) on \( \partial \mathcal{V} \). Note that for an unstable \( \theta \) the ratio would be expected to behave like \( \exp(c \lambda^{-2} |\mathcal{V}|) \). Part of the problem is figuring out which phases are stable since those are the only ones for which (1) will be true.

My proof of (1) begins with a cluster expansion - the GJS mean field expansion [3]. In each square, sum over the \( r \) possible wells for the average field to be near. The result is a Peierls expansion:

\[
\mathbb{Z}(\mathbb{V}^2) = \sum_{\Sigma} \mathbb{Z}_\Sigma(\mathbb{V}^2).
\]

Here \( \Sigma \) is a spin configuration assigning a well to each square in \( \mathcal{V} \). Wherever \( \Sigma \) specifies different wells in adjacent squares there will be a large fluctuation of the field, with correspondingly small probability of the order of \( \exp(-\lambda^{-2} |\Sigma|) \). Here \( |\Sigma| \) denotes the length of the Peierls contour for \( \Sigma \) - the curve across which \( \Sigma \) specifies...