SYMPLECTIC GEOMETRY AND QUANTISATION

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Introduction

In this talk I will describe a differential geometric formulation of canonical quantisation. These ideas, based on the symplectic geometry of the classical phase space, are largely due to B. Kostant and J.M. Souriau (independently) and were developed around 1964. There is an excellent comprehensive book on the topic by Woodhouse [2]. A good shorter account by Sniatycki [1] emphasises the computations relating to quantum mechanics. A recent paper by Woodhouse [3] shows how the ideas relate to quantum field theory.

The usual canonical quantisation proceeds by singling out on classical phase space a family of coordinate functions $p_1, q_1, ..., p_n, q_n$ in canonical pairs, and assigning operators to these and other relevant functions so that the operator commutator corresponds to the Poisson bracket $\{f, g\} = \sum (\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i})$.

For non-linear systems it may not be clear which choices of canonical variables $p_1, ..., p_n, q_1, ..., q_n$ will be appropriate, and indeed a globally valid system of canonical coordinates may not exist. It is therefore desirable to formulate the canonical quantisation in a coordinate-independent way and so that it applies to non-linear phase spaces.

We take as the fundamental data the classical phase space $M$ with its classical Poisson bracket structure. Thus we assume that local canonical coordinate systems $p_1, ..., p_n, q_1, ..., q_n$ exist on $M$ so that the differential 2-form $\omega = \sum dp_i \wedge dq_i$ has an invariant significance. This is equivalent to assuming that a 2-form $\omega$ is given on $M$ which is closed ($d\omega = 0$) and non-degenerate ($X \lrcorner \omega = 0$ for all $X$ if and only if $X = 0$). Here $X \lrcorner \omega$ denotes the contraction of $\omega$ with a vector field $X$. Such a 2-form $\omega$ is called a symplectic form. To each smooth function $H$ on $M$ we can then associate a Hamiltonian vector field $X_H$ defined by $dH + X_H \lrcorner \omega = 0$. The Poisson bracket of functions $H$ and $H'$ can be defined as

$$\{H, H'\} = \omega(X_H, X_{H'}) = X_H(H').$$

By working with the 2-form $\omega$, which has a global existence on the
phase space, we free ourselves from dependence on a choice of canonical variables, and are in effect working directly with the Poisson bracket. Thus we remain close to Dirac's correspondence principle.

The first step towards quantising is to choose a 'representation' in the sense of Dirac by choosing a 'maximal commuting set of classical observables'. If \( H_1, \ldots, H_n \) are independent Poisson commuting functions, \( \{H_j, H_k\} = 0 \), on an open set \( W \) then the Hamiltonian vector fields \( X_{H_1}, \ldots, X_{H_n} \) span an \( n \)-dimensional complex vector bundle over \( W \).

To obtain a corresponding global concept, we call a complex vector bundle \( F \), over a \( 2n \)-dimensional phase space \( M \), a polarisation if each point of \( M \) has a neighbourhood over which \( F \) is spanned by linearly independent Hamiltonian vector fields \( X_{H_1}, \ldots, X_{H_n} \) where \( H_1, \ldots, H_n \) are Poisson commuting. Thus we do not require the global existence of Poisson commuting functions to specify a polarisation. Equivalently, a polarisation may be defined as a vector bundle of complex tangent vectors of fibre dimension \( n \) such that the restriction of \( \omega \) to each fibre is zero. In the quantisation procedure to be described, a choice of polarisation effects a choice of quantum representation.

The quantum line bundle

For dynamical systems based on a configuration space with local position coordinates \( q^1, \ldots, q^n \) and associated momentum coordinates \( p_1, \ldots, p_n \), the 1-form \( pdq = \sum p_i dq^i \) plays an important and canonical role. It gives the Lagrangian associated with the Hamiltonian \( H \):

\[
\dot{q}^i = H X_{H_i} \quad \text{pdq} = -H.
\]

We also have \( d(pdq) = \omega \).

On a general phase space we may not have such a form 'pdq' globally, and in any case there may not be a canonical choice. We can however, since \( d\omega = 0 \), choose a 1-form \( \theta \) locally so that \( d\theta = \omega \). The symplectic potential \( \theta \) will be determined only up to a gauge transformation \( \theta \to \theta + du \). A choice of \( \theta \) may be regarded as fixing a gauge and fixing a corresponding Lagrangian \( X_H \theta = H \) locally.

In order to quantise the system we fix a polarisation \( F \). The quantum mechanical wave functions in the \( F \)-representation, and in the gauge \( \theta \), are then defined to be the complex valued functions \( \phi \) on \( M \) such that

\[
(X - \frac{1}{i} X \theta) \phi = 0
\]

for all vector fields \( X \) on \( M \) which take their values in the selected