Estimating a Probability Using Finite Memory *

Extended Abstract

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Abstract: Let \( \{ i \} \) be a sequence of independent Bernoulli random variables with probability \( p \) that \( X_i = 1 \) and probability \( q = 1 - p \) that \( X_i = 0 \) for all \( i \geq 1 \). We consider time-invariant finite-memory (i.e., finite-state) estimation procedures for the parameter \( p \) which take \( X_1, \ldots \) as an input sequence. In particular, we describe an \( n \)-state deterministic estimation procedure that can estimate \( p \) with mean-square error \( O(\frac{\log n}{n}) \) and an \( n \)-state probabilistic estimation procedure that can estimate \( p \) with mean-square error \( O(\frac{1}{n}) \). We prove that the \( O(\frac{1}{n}) \) bound is optimal up to within a constant factor. In addition, we show that linear estimation procedures are just as powerful (up to the measure of mean-square error) as arbitrary estimation procedures. The proofs are based on the Markov Chain Tree Theorem.

1. Introduction

Let \( \{ X_i \} \) be a sequence of independent Bernoulli random variables with probability \( p \) that \( X_i = 1 \) and probability \( q = 1 - p \) that \( X_i = 0 \) for all \( i \geq 1 \). Estimating the value of \( p \) is a classical problem in statistics. In general, an estimation procedure for \( p \) consists of a sequence of estimates \( \{ e_t \} \) where each \( e_t \) is a function of \( \{ X_i \} \). When the form of the estimation procedure is unrestricted, it is well-known that \( p \) is best estimated by

\[
e_t = \frac{1}{t} \sum_{i=1}^{t} X_i.
\]

As an example, consider the problem of estimating the probability \( p \) that a coin of unknown bias will come up "heads". The optimal estimation procedure will, on the \( t \)th trial, flip the coin to determine \( X_t \) (\( X_t = 1 \) for "heads" and \( X_t = 0 \) for "tails") and then estimate the proportion of heads observed in the first \( t \) trials.

The quality of an estimation procedure may be measured by its mean-square error \( \sigma^2(p) \). The mean-square error of an estimation procedure is defined as

\[
\sigma^2(p) = \lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} \sigma_i^2(p),
\]

where

\[
\sigma_i^2(p) = E((e_i - p)^2)
\]

denotes the expected square error of the \( i \)th estimate. For example, it is well-known that \( \sigma_i^2(p) = \frac{pq}{t} \) and \( \sigma^2(p) = 0 \) when \( e_i = \frac{1}{t} \sum_{i=1}^{t} X_i \).

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In this paper, we consider time-invariant estimation procedures which are restricted to use a finite amount of memory. A *time-invariant finite-memory estimation procedure* consists of a finite number of states \( S = \{1, \ldots, n\} \), a start state \( S_0 \in \{1, \ldots, n\} \), and a transition function \( \tau \) which computes the state \( S_t \) at step \( t \) from the state \( S_{t-1} \) at step \( t-1 \) and the input \( X_t \) according to

\[
S_t = \tau(S_{t-1}, X_t).
\]

In addition, each state \( i \) is associated with an estimate \( \eta_i \) of \( p \). The estimate after the \( t \)th transition is then given by \( e_t = \eta_{S_t} \). For simplicity, we will call a finite-state estimation procedure an "FSE".

As an example, consider the FSE shown in Figure 1. This FSE has \( n = \frac{(s+1)(s+2)}{2} \) states and simulates two counters: one for the number of inputs seen, and one for the number of inputs seen that are ones. Because of the finite-state restriction, the counters can count up to \( s = \Theta(\sqrt{n}) \) but not beyond. Hence, all inputs after the \( s \)th input are ignored. On the \( t \)th step, the FSE estimates the proportion of ones seen in the first \( \min(s, t) \) inputs. This is

\[
e_t = \frac{1}{\min(s, t)} \sum_{i=1}^{\min(s, t)} X_i.
\]

Hence the mean-square error of the FSE is \( \sigma^2(p) = \frac{pq}{s} = O\left(\frac{1}{\sqrt{n}}\right) \).

**Figure 1**: An \( \frac{(s+1)(s+2)}{2} \)-state deterministic FSE with mean-square error \( \sigma^2(p) = \frac{pq}{s} \). States are represented by circles. Arrows labeled with \( q \) denote transitions on input zero. Arrows labeled with \( p \) denote transitions on input one. Estimates are given as fractions and represent the proportion of inputs seen that are ones.