MODELLING OF TWO-DIMENSIONAL BUBBLES IN VERTICAL TUBES

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INTRODUCTION

A problem related to gas-liquid slug flow in tubes arises in understanding the dynamics of the so-called Taylor bubbles preceding the slug. These bubbles occupy a significant cross-section of the tube and are often many tube-radii long. While for vertical flow the problem can be considered axisymmetric, we will consider a slightly simpler problem, namely, the flow past a two-dimensional bubble between two flat plates. As in the slug flow problem, the fluid runs around the outside of the bubble and remains as a layer down the surface of the tube falling freely under gravity. A characteristic quantity for this flow is the Froude number $F_r = \frac{U}{\sqrt{gb}}$, a dimensionless measure of the velocity $U$ with which the gas bubble penetrates the liquid, $g$ is the gravitational acceleration and $b$ the two-dimensional tube's half width. The Froude number can be associated with the Reynolds number in the tube by the following relationship,

$$R_e = \frac{2Ub}{\nu} = \left( \frac{2b^{3/2} \left( \frac{g^{1/2}}{\nu} \right)}{F_r} \right) F_r,$$

where $\nu$ is the kinematic viscosity of the liquid. For notation, we let $x$ represent the vertical distance from the bubble nose, $y$ the horizontal distance from the centerline, and the vector $\mathbf{r} = (x, y)$. We begin by computing, for each Froude number, the bubble shape needed to achieve a constant pressure along its surface in an inviscid, irrotational flow. A turbulent boundary-layer analysis is then applied in the region between the bubble and the tube. Results on the liquid film thickness at the intersection of the developing boundary layer and the bubble surface are presented.

POTENTIAL ANALYSIS

The irrotational flow field can be described by a stream function $\psi$ written as a boundary integral of singularity distributions:

$$\psi(\mathbf{r}) = -\frac{1}{4\pi} \oint_C \gamma(\mathbf{r}') \ln \left( \frac{1}{|\mathbf{r}-\mathbf{r}'|^2} \right) \, ds'.$$

The boundary $C$ of the fluid domain contained between the tube wall and the bubble surface is divided into $N$ segments with $\gamma$ constant on each segment and represented by $N$ dipoles, each positioned at a given $\mathbf{r}$. The dipole strengths are obtained as the solution of the set of linear simultaneous equations:

$$\psi(\mathbf{r}_j) = -\frac{1}{4\pi} \sum_{i=1}^{N} \gamma(\mathbf{r}_i) \ln \left( \sigma + |\mathbf{r}_i-\mathbf{r}_j|^2 \right), \quad j = 1, \ldots, N,$$

where $\sigma$ is a small non-vanishing core radius for the dipoles and is included to handle the logarithmic singularity. It is assumed that upstream of the bubble the flow comes in uniformly. Similarly, for the outflow condition of the domain, the vertical (streamwise) component of the velocity is approximated to be a constant across the thin film. The stream function $\psi$ is therefore prescribed as being a linear function.
of \( y \) for the inflow and outflow conditions while it is a constant along the tube \( (y = \pm b) \) and zero along the free surface of the bubble. An initial set of positions is picked for the dipoles describing the bubble and the rest of the boundary curve \( C \) is discretized using a well defined set of dipoles. Solving the linear system of equations (3) then provides their corresponding strengths which, in turn, give the velocity of the liquid along the surface of the bubble. Through Bernoulli's equation, a pressure is obtained which, if not zero, calls for an upgrade of the positions of the dipoles on the bubble. Iterating in this fashion finally provides the zero-pressure curve that describes the bubble.

It should be pointed out that the matrix associated with the linear system (3) is full. With about 200 dipoles to describe a bubble of a length equal up to 14 tube radii and another 300 dipoles to describe the rest of the boundary curve, \( N = 500 \). Since the system of equations (3) has to be solved for every non-linear iteration, it obviously turns out to be the costliest part of the computational effort. By observing that the matrix for the logarithmic coefficients of (3) is symmetric and that it does not change very much from a suitable initial guess for the bubble to the zero-pressure final curve, a preconditioned iterative technique to solve the system was used in lieu of the direct method (Cholesky factorization and back substitution). A conjugate gradient technique was the most convenient to implement. The preconditioning approach consists of premultiplying the system by an approximate inverse to the coefficient matrix so that the spectrum of the resulting matrix is closer to unity and thereby accelerate the convergence of the conjugate gradient technique. This inverse was determined and updated by using the direct method every tenth iteration. In that fashion, the computing time was cut by at least a factor of 4 from the direct method used alone.

The results of the computations appear in Figure 1 which displays some of the shapes satisfying the zero-pressure condition at the surface. Each shape is characterized by a particular Froude number. Garabedian (1957) also showed that the velocity of steady flow past a bubble in an infinitely long tube is not fixed by the tube diameter and the acceleration of gravity alone. Collins (1965) and Maneri (1970) conducted experiments on two-dimensional bubbles (between parallel plates of high aspect ratio so as to approximate a two-dimensional flow). They both observed that the bubbles have a circular shape at the nose. If we choose to satisfy this criterion in our potential flow simulations, a particular profile can now be selected from the family of curves. The computed result is that \( F_r = 0.32 \) and the ratio of the radius of curvature to the radius of the tube equals 0.64. This compared with 0.32 and 0.62 reported by Collins (1965) while Maneri (1970) measured values of 0.36 and 0.64 for parallel plates of the largest aspect ratio used. Figure 2 shows our calculated profile (solid line) compared with the profile obtained by digitization of Maneri's observation (dashed line). The agreement is very good indeed.

**BOUNDARY-LAYER ANALYSIS**

Although an inviscid approach is reasonably correct near the front of the bubble, it must break down further from the nose since it requires the surface to asymptote to the outer wall as \( x \to \infty \) while observations seem to indicate an asymptotic, nonzero thickness well before then. This equilibrium thickness is conjectured to be the result of viscous forces acting in the thin liquid layer. To examine its effect, we will consider the development of a turbulent boundary layer between the bubble and the wall and its relation to a free falling film.

Taking a bubble rising with velocity \( U \) in a tube of radius \( b \), we let the thickness of the liquid film at a distance \( x \) from the nose be \( \delta(x) \) (see Figure 3). We will assume that the boundary layer begins at \( x = 0 \) and that its thickness is given by \( \delta(x) \). Its subsequent development downstream is determined by the Von Karman momentum integral equation (Schlichting 1979),

\[
\int_0^\delta \frac{d}{dx} (u(x_0 - u)) \, dy' + \frac{du_0}{dx} \int_0^\delta (u(x_0 - u)) \, dy' = \frac{\tau_w}{\rho}.
\]