PART III

AUTOMATA ON INFINITE TREES
In this article we give an outline of a new theory of automata on
infinite trees (or indeed, finite trees) which allows us to give a simple proof
of Rabin's Theorem that the monadic theory of the binary tree is decidable. The
concept of an alternating automaton described by Chandra, Kozen and
Stockmeyer [1] generalizes the notion of nondeterminism by allowing states to be
existential, universal or negating. We present a different notion of an alternating
automaton in which the transition function is a homomorphism into a free
distributive lattice. The availability of the lattice operations makes complementation
easy, even for infinite trees. Given an automaton $M$ which is alternating in our
sense, one can define the dual automaton $M$ which always accepts the
complement of the language accepted by $M$. The proof of this fact directly uses
the determinacy of certain infinite games and does not require an effective
version of determinacy. It is not immediately obvious that the alternating automata
which we define are equivalent to ordinary nondeterministic automata on infinite
trees. We need this result in order to prove Rabin’s theorem. To establish the
result we prove a “uniformization theorem” which shows that certain
nondeterministic automata have a “sufficiently uniform” strategy for acceptance.
Gurevich and Harrington [2] have recently given a new proof of Rabin’s theorem
in which they do not reformulate the automata theory but instead prove that a
special form of “forgetful determinacy” holds for a game associated directly with
a nondeterministic automaton. The result which we prove is weaker in that the
choices in the strategy may depend on an unbounded amount of information about
the past history. This allows the strategy to be simply defined and verified.

One result of our method is that there is an effective construction
which associates with each sentence $\phi$ of the monadic theory of the $k$-ary tree
a sentence $\phi'$ of the monadic theory of the natural numbers $\mathbb{N}$ such that $\phi$
is true if and only if $\phi'$ is true. A relativized version of this result gives a
proper extension of the monadic theory of the binary tree which is still
decidable. Finally, since the proof of the complementation theorem depends only
on the fact that the acceptance condition is Borel, the theory raises interesting
questions concerning the possibility of using other acceptance conditions.

Our work has benefited greatly from many conversations with Ward
Henson, André Muchnik and Alexei Simenov. Our proof of the uniformization
theorem uses Muchnik’s ingenious idea of decomposing the behaviour of a
machine into the behaviours of the machines obtained by respectively converting
each state into a dead-end marker and we learned this idea from Muchnik and
Simenov. The influence of Ward Henson is felt throughout.

2. Alternating Automata on Strings and Trees

We define our notion of an alternating automaton and the definition of
acceptance. Although it is the case of trees which really shows the naturalness
of the lattice formulation, we begin by considering automata on finite strings. In
a nondeterministic automaton the transition function, for each input letter, maps
the present state to a collection of states. One member of this collection is
thought of as being chosen nondeterministically. An input string is accepted if it
is possible to make a sequence of nondeterministic choices which lead to a final
state.