Sedimentation of a Random Dilute Suspension

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ABSTRACT

Recently Batchelor has calculated the average sedimentation speed of a random dilute suspension. A description of that work is presented here, along with a computation of the variance in the speed. It is shown that under Batchelor's assumptions and in the appropriate continuum limit, the variance is infinite.

1. Introduction

The motion of a solid spherical particle through a viscous fluid under the effect of gravity can be described by Stokes equations if the Reynolds number $\text{Re} = a |\bar{v}_{ST}| \rho_p / \mu$ is sufficiently small. In this formula the Stokes speed is

$$\bar{v}_{ST} = - \hat{e} m \bar{g} / 6 \pi \mu a \quad (1)$$

and $a$ is the particle radius, $m$ is the particle mass, $\rho_p$ and $\rho_f$ are the particle and fluid densities, $\mu$ is the fluid viscosity, $\bar{g} = g (1 - \rho_f / \rho_p)$ is the buoyancy-reduced gravitational acceleration and $\hat{e}$ is the unit vertical vector. Since $\rho_p < \rho_f$, the Reynolds number based on particle density $\rho_p$ has been used so that particle inertia and fluid inertia are both negligible.

Consider a suspension of $N$ of these identical particles which are randomly distributed throughout a fixed container of volume $V$. The average number density of particles is $n = N/|V|$ and the volume fraction is $\beta = \frac{4}{3} \pi a^3 n$. The suspension may be expected to behave as a continuum if $n$ is large, i.e. in the limit

$$N \rightarrow \infty \text{ with } \beta \text{ constant.} \quad (2)$$

For a dilute suspension with $\beta \ll 1$, the dependence of the sedimentation speed $\bar{v}_S$ has been analyzed in several different cases. For a periodic array of spheres, Hasimoto [4] found $\bar{v}_S$ to be

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\[ \tilde{v}_s = \tilde{v}_{ST} (1 - c \beta^{1/3} + O(\beta)) \] (3)

in which the positive constant \( c \) depends on the type of periodic lattice. For a uniformly distributed random suspension Batchelor [1] showed that

\[ \tilde{v}_s = \tilde{v}_{ST} (1 - 6.55 \beta + O(\beta^2)). \] (4)

These results are important for prediction of the behavior of dilute suspensions, as well as for indication of the effect of higher volume fractions. However the correct particle distribution - uniformly random, periodic, or something in between - is uncertain.

In this paper the results of Batchelor for a random suspension are derived and discussed and the variance in the sedimentation speed is calculated for such a suspension.

2. Stokes Equations.

Stokes equations, for the motion of a fluid in which inertial effects are negligible, are

\[ \mu \nabla \vec{u} - \nabla p = 0 \] (5)
\[ \nabla \cdot \vec{u} = 0. \] (6)

If the fluid contains \( N \) particles with centers \( \vec{x}_1, \ldots, \vec{x}_N \) each of radius \( a \), then equations (5) and (6) hold for \( \vec{x} \in V \) with \( |\vec{x} - \vec{x}_i| > a \) for all \( i \). In addition the conditions of no slip on the particle surface and balance of forces on the particle (neglecting inertia) are (for \( I = 1, \ldots, N \))

\[ \vec{u}(x) = \vec{v}_i \text{ on } |\vec{x} - \vec{x}_i| = a \] (7)
\[ \vec{F}_i = \int_{|\vec{x} - \vec{x}_i| = a} \sigma \cdot \hat{n} \, ds \] (8)

in which \( \vec{v}_i \) is the particle velocity and is independent of \( \vec{x} \), \( \vec{F}_i \) is the external force on the particle, \( \hat{n} \) is the outward normal to the sphere and \( \sigma_{ij} = -\rho \delta_{ij} + 2\mu (\partial u_i/\partial x_j + \partial u_j/\partial x_i) \) is the stress tensor in the fluid.

In these equations particle rotation has been artificially set to zero since its effect on sedimentation is negligible. For the case in which the forces \( \vec{F}_i \) are prescribed, the velocities \( \vec{v}_i \) must be determined as part of the problem. On the other hand if the velocities \( \vec{v}_i \) are prescribed, equation (8) is superfluous.

The subsequent developments in this paper rely on Faxen’s law, which relates the solution of Stokes equations with a single particle to the solution without any particles. Its statement is the following: